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GAUGE THEORIES IN FINITE VOLUMES REVISITED

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Hamiltonian perturbation theory to fourth order in the gauge-coupling constant is applied to compute Lüscher's effective hamiltonian for the zero-momentum gauge fields in a finite cubic volume in order to understand some apparent gauge dependence at the two-loop level. This allows us to confirm earlier results at a more rigorous level and to include new terms that mix coordinate and momentum operators, previously ignored. After that we live up to our promise to give some of the details for the derivation of the effective hamiltonian starting from Wilson's lattice action, which allowed a semi-analytic study of lattice artifacts. We also discuss some issues related to two-loop lattice perturbation theory. For easy reference the continuum effective hamiltonian and the lattice effective lagrangian, together with the specification of the boundary conditions and the numerical values of the coefficients, are summarized in a separate section.

1. Introduction

In this paper we will analyse in more rigour and detail the finite-volume expansion of SU(2) gauge theories on a torus. One of the most useful applications of the finite-volume expansion turned out to be a detailed comparison with lattice Monte Carlo calculations. Since the numerical accuracy and reliability of the computer simulations have improved considerably over the past few years [1, 2], systematic differences showed up with the finite-volume expansion in the continuum [3]. It was most natural to associate those differences with the effect of a finite lattice. The results [4] of the computation, to be described in more detail here, did indeed confirm that the observed deviation from the continuum result and the spread of the Monte Carlo data is mainly due to lattice artifacts.

The finite-volume expansion is based on computing an effective hamiltonian for the zero-momentum gauge fields, as first derived by Lüscher [5]. In lowest order this is just the Yang–Mills hamiltonian, truncated to these zero-momentum gauge

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fields, with the bare gauge-coupling constant replaced by the renormalized one

$$H^{(0)} = -\frac{g^2}{2L}\frac{\partial^2}{\partial c_i^{a2}} + \frac{1}{4g^2L}\left(\varepsilon_{abd}c_i^a c_j^b\right)^2, \tag{1.1}$$

where c_i^a is the (rescaled) zero-momentum gauge field

$$A_i^a(x) = c_i^a/L. \tag{1.2}$$

In the continuum, hamiltonian perturbation theory [5] will be applied as a reliable means of deriving the effective hamiltonian for the zero-momentum modes to fourth order in the coupling constant, which involves a two-loop computation. The main new result in this paper is to provide the complete effective hamiltonian to that order, which includes terms of the form $c^2\partial^2/\partial c^2$ not accessible in a background field analysis [6]. Apart from these terms, the resulting effective hamiltonian can be represented (with some good fortune) in a form identical to what was derived from the background field analysis. On the lattice we will derive the effective lagrangian in the zero-momentum modes, using the one-loop background field analysis. This yields a gauge theory on a lattice that has one link in each of the spatial directions and is infinite in the time direction. We also analyse in detail how to obtain the effective hamiltonian from the transfer matrix defined by the effective lattice model. The spectrum of this hamiltonian can then be obtained exactly as in the continuum [6], using a standard Rayleigh–Ritz variational analysis. Readers mainly interested in the results are advised to consult sect. 8.

The main mechanism behind the dynamics in intermediate volumes is due to incorporating non-perturbative effects associated to the Gribov horizon, or equivalently, due to the topological non-trivial nature of configuration space. We have discussed these issues at great length elsewhere [6]. Here we simply state that as a consequence of the topological non-triviality, the configuration space for the zero-momentum gauge fields is compact and can to a good accuracy be divided (for SU(2)) in eight coordinate patches with transition functions essentially described by gauge transformations. It can alternatively be formulated by defining the theory on one of these patches, with appropriate boundary conditions on the wave function $\psi(c)$

$$\psi(c) = 0, \qquad \frac{\partial}{\partial r_i} r_i \psi(c) = 0, \qquad \text{at } r_i = \pi, \tag{1.3}$$

where

$$r_i = \sqrt{\sum_{a=1}^{3} c_i^a c_i^a}\,. \tag{1.4}$$

The appropriate choice of boundary conditions is determined by the quantum numbers of the particular state one wishes to consider [6, 7]. For a cubic volume these are labelled by the irreducible representation of the cubic group $\{A_1^\pm, A_2^\pm, E^\pm, T_1^\pm, T_2^\pm\}$ and 't Hooft's electric flux [8] $\{e_i = \pm 1\}$. Sect. 8 will review which boundary conditions are associated to each of these states (with positive parity).

At small volumes (or equivalently due to asymptotic freedom, at small coupling), there is a quantum-induced potential barrier that separates the different coordinate patches and the boundary conditions are irrelevant. As a consequence, electric flux energies are exponentially suppressed. At intermediate volumes the coupling is so strong that the potential barrier forms no obstacle and the boundary conditions strongly dominate the dynamics. At yet larger volumes classical barriers which separate coordinate patches, whose overlap was not yet taken into account will no longer be obstacles either, but these effects cannot be incorporated within a framework of a zero-momentum effective hamiltonian. From the comparison with Monte Carlo results and theoretical arguments this occurs at volumes larger than about 0.7 fermi (with the physical scale set by a string tension of 420 MeV^2). This is where for the tensor glueball one observes restoration of rotational symmetry [2, 3]. It is also the distance beyond which the topological susceptibility seems to suddenly switch on [9].

It is clear that the accuracy of the results in the accessible intermediate volume range is also determined by how accurate the effective hamiltonian can be computed. This in itself is a perturbative computation, which is strictly separated from the issue of the non-perturbative effects incorporated by imposing the boundary conditions. Lüscher computed the effective hamiltonian up to the order $O(g^{8/3})$, where $c = O(g^{2/3})$ [5]. This involves only a one-loop computation and can be obtained in three different ways. The first is using degenerate (Bloch [10]) hamiltonian perturbation theory, the second [11]* is by calculating the euclidean transition function from one to another zero-momentum gauge field and reading off the effective hamiltonian that reproduces this transition function. The third method uses a background field type calculation with the non-local gauge [6]

$$\chi_N = (1-P)\,\partial_\mu A_\mu + i\left[PA_\mu, A_\mu\right], \tag{1.5}$$

where $A_\mu = A_\mu^a \sigma_a/2$ (σ_a are the Pauli matrices) and

$$PA_\mu = L^{-3}\int \mathrm{d}^3x A_\mu(x) \tag{1.6}$$

is the projection on the zero-momentum gauge field. Unlike in the second method

* The last two references apply this method to the $O(N)$ model.

one does not integrate over the zero-momentum modes. However, in principle one needs to determine the effective action for an arbitrary time-dependent background field $c(t)$. Nevertheless, after a simple rescaling, taking $c(t)$ a time-independent background field did reproduce to $O(g^{8/3})$ the correct effective potential, and this was by far computationally the simplest method. We will address the validity of this approximation by carefully considering issues concerning gauge dependence and contributions at two-loop order.

The remainder of this paper is organized as follows. In sect. 2 we will summarise problems we encountered by reconsidering the two-loop contribution to the effective potential when using a time-independent background field. We also analyse some issues concerning gauge dependence. Since, the two-loop corrections contribute up to 3% to especially the energy of electric flux, it is imperative to perform the calculations of the effective hamiltonian using a more systematic method. For this we have used hamiltonian perturbation theory to $O(g^4)$ described in sect. 3. These calculations fortunately will confirm our earlier results [6]. It is also worthwhile noting that the way we dealt with the Gribov problem uses in an essential way the hamiltonian formulation. It is thus gratifying, from the point of consistency, to also have computed the effective hamiltonian using the same formalism. In sect. 4 we derive the effective action starting from Wilson's lattice action, using the one-loop background field technique. It was our attempt to also find the two-loop lattice corrections that made us aware of the problems discussed in sect. 2. Sect. 5 describes how the effective action, which defines a transfer matrix, can be used to determine the effective hamiltonian, thereby including some important additional lattice artifacts. Sect. 6 will be dedicated to a discussion of some aspects of lattice perturbation theory at the two-loop level for a time-independent abelian background field. As we have argued, this calculation does not give the complete result for the effective potential. Nevertheless, the calculation is set up so as to be applicable in more general situations. In particular one can obtain the lattice vacuum energies in the presence of a magnetic flux, imposed by twisted boundary conditions [8], in a straightforward manner from the results we present. Sect. 7 provides a discussion. Sect. 8 concludes the paper with a summary of the results for easy reference. There we list the continuum effective hamiltonian and the lattice effective lagrangian in a finite cubic volume, we supply the numerical values of the various coefficients and review the assignment of the appropriate boundary conditions. We also present a formulation of the effective lattice theory that can be analysed using the Monte Carlo method.

2. Trouble with the background field

We will consider the background field method described in the introduction for both the non-local gauge (eq. (5)) and for the Lorentz gauge $\chi_L = (1-P)\partial_\mu A_\mu$.

These gauge conditions are invariant under spatially constant gauge transformations, which are the symmetries of the effective lagrangian and in the background field calculation we only integrate over the non-zero momentum gauge and ghost fields. To $O(g^4)$ it is straightforward [6] to compute the one-loop effective potential for time-independent background fields (d is the space dimension)

$$V_1(c) = L^{-1}\left\{\gamma_1 \sum_i r_i^2 + \gamma_2 \sum_i r_i^4 + \gamma_3 \sum_{i<j} r_i^2 r_j^2 + \gamma_4 \sum_i r_i^6 + \gamma_5 \sum_{i\neq j} r_i^2 r_j^4 + \gamma_6 \prod_i r_i^2 \right.$$
$$\left. + \frac{1}{4}\left(\frac{L^{d-3}}{g_0^2} + \alpha_2\right) \sum_{i,j} F_{ij}^2 + \alpha_3 \sum_{i,j,k} F_{ij}^2 r_k^2 + \alpha_4 \sum_{i\neq j} F_{ij}^2 r_j^2 + \alpha_5 \det{}^2(c) \right\} + V_1(0),$$
$$V_1(0) = 3L^{-1} \sum k = -48\pi^2 L^{-1} \sum k^{-4}. \tag{2.1}$$

Table 1 contains the coefficients in terms of momentum sums for the two gauges. By taking appropriate factors of L out, we take the momenta dimensionless, thus $\boldsymbol{k} \in 2\pi\mathbb{Z}^d$ and we use the convention $k = |\boldsymbol{k}|$ for the length of a momentum vector. We have also included the coefficient α_1, which gives the one-loop correction to the kinetic term

$$\frac{L}{2}\left(\frac{L^{d-3}}{g_0^2} + \alpha_1\right)\left(\frac{dc(t)}{dt}\right)^2 .$$

We note that all the coefficients for the terms which vanish when the background field satisfies the equations of motion (i.e. those terms which vanish for abelian background fields) depend on the gauge, but that the difference can be absorbed by a rescaling of the fields

$$c_i^a \to c_i^a - \tfrac{1}{2} g_0^2 c_i^a \sum k^{-3} + \tfrac{25}{96} g_0^2 c_i^a c_j^b c_j^b \sum k^{-5} - \tfrac{19}{32} g_0^2 c_i^b c_j^a c_j^b \sum k^{-5}, \tag{2.2}$$

which transforms the results for the Lorentz gauge to those in the non-local gauge.

We now address the two-loop contributions to the effective potential for time-independent background fields. For the non-local gauge this goes back to the calculation of the two-loop vacuum energy [12]. In particular, having verified that to $O(g^2)$ the one-loop effective lagrangian is gauge independent, we should insist that the two-loop vacuum energy is likewise gauge independent. This turns out not to be the case if we restrict to time-independent background fields. To be more precise, we take a vanishing background field to get the two-loop $O(g^2)$ contribution. The Feynman rules in the Lorentz gauge are exactly those of ordinary SU(2) gauge theory, with the only difference that we sum (rather than integrate) over the (non-zero) spatial momenta. The two-loop vacuum energy in the Lorentz gauge is

TABLE 1

The coefficients as occurring in the one-loop effective potential, eq. (2.1), in the approximation of a time-independent background field for the non-local gauge (1.5), Lorentz gauge (1.6) and the Coulomb gauge in the hamiltonian formulation of sect. 3. The coefficients are to be summed over all non-zero momenta, where $\boldsymbol{k} \in (2\pi\mathbb{Z})^3$ and $k \equiv |\boldsymbol{k}|$. When the momentum sums are divergent in three dimensions we give the coefficients as a function of d. (Note that $\gamma_{1,2,3}$ have a finite $d \to 3$ limit and that all γ_i are gauge independent)

	Non-local gauge	Lorentz gauge	Coulomb gauge ($H'_{(2)}$)
$\alpha_1(d)$	$-\dfrac{7d+1}{4dk^3}$	$-\dfrac{3d+1}{4dk^3}$	$-\dfrac{d-1}{dk^3}$
$\alpha_2(d)$	$\dfrac{d^2-15d+6}{4dk^3}+\dfrac{5(d-1)k_1^2k_2^2}{k^7}$	$\dfrac{d^2-7d+6}{4dk^3}+\dfrac{5(d-1)k_1^2k_2^2}{k^7}$	$\dfrac{d^2-9d+16}{4dk^3}+\dfrac{5(d-1)k_1^2k_2^2}{k^7}$
α_3	$\dfrac{583}{96k^5}-\dfrac{1029k_1^4}{16k^9}+\dfrac{189k_1^6}{4k^{11}}$	$\dfrac{391}{64k^5}-\dfrac{1029k_1^4}{16k^9}+\dfrac{189k_1^6}{4k^{11}}$	$\dfrac{599}{96k^5}-\dfrac{1025k_1^4}{16k^9}+\dfrac{189k_1^6}{4k^{11}}$
α_4	$-\dfrac{469}{32k^5}+\dfrac{4865k_1^4}{32k^9}-\dfrac{441k_1^6}{4k^{11}}$	$-\dfrac{469}{32k^5}+\dfrac{4865k_1^4}{32k^9}-\dfrac{441k_1^6}{4k^{11}}$	$-\dfrac{465}{32k^5}+\dfrac{4845k_1^4}{32k^9}-\dfrac{441k_1^6}{4k^{11}}$
α_5	$-\dfrac{39}{4k^5}+\dfrac{1407k_1^4}{16k^9}-\dfrac{63k_1^6}{k^{11}}$	$-\dfrac{369}{32k^5}+\dfrac{1407k_1^4}{16k^9}-\dfrac{63k_1^6}{k^{11}}$	$-\dfrac{23}{2k^5}+\dfrac{1395k_1^4}{16k^9}-\dfrac{63k_1^6}{k^{11}}$
$\gamma_1(d)$	$\dfrac{(d-1)^2}{2dk}$		
$\gamma_2(d)$	$-\dfrac{(d-1)(d-6)}{8dk^3}-\dfrac{5(d-1)k_1^4}{8k^7}$		
$\gamma_3(d)$	$-\dfrac{(d-1)(d-6)}{4dk^3}-\dfrac{15(d-1)k_1^4}{4k^7}$		
γ_4	$-\dfrac{1}{2k^5}+\dfrac{35k_1^4}{8k^9}-\dfrac{21k_1^6}{8k^{11}}$		
γ_5	$\dfrac{23}{8k^5}-\dfrac{445k_1^4}{16k^9}+\dfrac{315k_1^6}{16k^{11}}$		
γ_6	$-\dfrac{117}{4k^5}+\dfrac{315k_1^4}{k^9}-\dfrac{945k_1^6}{4k^{11}}$		

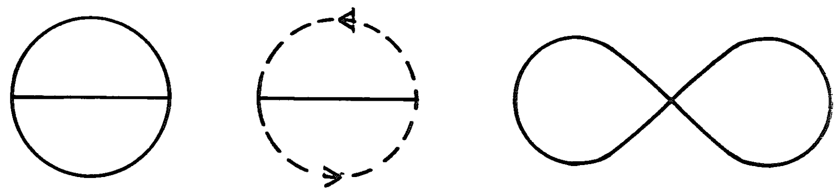


Fig. 1. The two-loop Feynman diagrams that contribute to V_2.

then obtained by summing the three diagrams in fig. 1, which after some simple analysis (see eq. (19) of ref. [12]) yields (usually we will ignore terms that vanish as $d \to 3$)

$$V_2^{(L)}(0) = -\frac{27g_0^2}{8L}\sum_{k+p\neq 0} k^{-1}p^{-1} + \frac{3g_0^2}{8L}\sum_{k+p\neq 0} k^{-1}p^{-1} + \frac{9g_0^2}{2L}\left(\sum k^{-1}\right)^2$$

$$= \frac{3g_0^2}{2L}\left(\sum k^{-1}\right)^2 + \frac{3g_0^2}{L}\sum k^{-2}. \tag{2.3}$$

The subsequent terms in the first identity for $V_2^{(L)}(0)$ correspond to the subsequent diagrams of fig. 1. For the non-local gauge we get exactly the same contributions. except for an additional "non-local" term coming from the Faddeev–Popov determinant. It was claimed in ref. [12] that this contribution exactly cancels against the term proportional to $\sum k^{-2}$ in eq. (2.3). Actually, when we worked out the two-loop Feynman graphs on the lattice, we discovered that the "non-local" vertex associated to the ghosts in eq. (22) of ref. [12] was wrong by a factor of -2, such that now there is no cancellation. To have an independent check it is useful to calculate the Faddeev–Popov determinant directly without using ghosts. We find under an infinitesimal gauge transformation

$$\delta\chi_N = (1-P)\partial_\mu D_\mu\Lambda + i[iP[A_\mu,\Lambda],A_\mu] = \left(\partial_\mu^2 + ig_0\partial_\mu \operatorname{ad} Q_\mu + g_0^2 \operatorname{ad} Q_\mu P \operatorname{ad} Q_\mu\right)\Lambda, \tag{2.4}$$

where we used that the background field vanishes and thus $A_\mu = g_0 Q_\mu$ has non-zero momentum. The Faddeev–Popov determinant therefore contributes

$$iT^{-1}\left\langle \operatorname{Tr}\ln\left(1 + ig_0\partial_\mu \operatorname{ad} Q_\mu \partial_\nu^{-2} + g_0^2 \operatorname{ad} Q_\mu P \operatorname{ad} Q_\mu \partial_\nu^{-2}\right)\right\rangle_c \tag{2.5}$$

to the effective potential. The expectation value is with respect to the gauge fields.

After performing a Wick rotation one easily finds this to equal

$$-\frac{2g_0^2}{TL^3}\sum_{k+p=0}\frac{\langle Q_\mu^a(-p)Q_\mu^a(p)\rangle}{k_\nu^2}$$

$$-\frac{g_0^2}{TL^3}\sum_{k-p\neq 0}\frac{\langle k_\mu Q_\mu^a(k-p)p_\nu Q_\nu^a(p-k)\rangle}{k_\lambda^2 p_\rho^2}+\mathrm{O}(g_0^4). \tag{2.6}$$

Together with $\langle Q_\mu^a(-p)Q_\nu^b(p)\rangle=\delta_{\mu\nu}\delta_{ab}/p_\lambda^2$ this gives for the non-local gauge the following contribution of the Faddeev–Popov determinant tc the two-loop vacuum energy

$$V_2^{(N)}(0)_{\mathrm{FP}}=\frac{3g_0^2}{8L}\sum_{k+p\neq 0}k^{-1}p^{-1}+\frac{6g_0^2}{L}\sum k^{-2}. \tag{2.7}$$

The first term corresponds to the second term in eq. (2.3), the second term is due to the "non-local" vertex. The value of $V_2(0)$ does not actually influence the mass spectrum, it is therefore useful to also compute the next field-dependent term in the two-loop effective potential (which is proportional to $g_0^2c^2/L$). For the non-local gauge χ_N we can use eqs. (2.25) and (2.26) of ref. [6], corrected for the proper non-local four-point vertex. Alternatively one can for both gauges calculate the three- and four-point vertices as a function of the background field and expand the background field dependent propagator up to $\mathrm{O}(c^2)$. One finds, together with the non-local vertex, which contributes $g_0^2L^{-1}\{6\sum k^{-2}+\frac{2}{3}\sum k^{-4}\}$, for the non-local gauge

$$V_2^{(N)}(c)=\frac{g_0^2}{L}\left\{\tfrac{3}{2}\left(\sum k^{-1}\right)^2+9\sum k^{-2}\right\}+\frac{g_0^2c^2}{L}\left\{-\frac{(d-3)}{3}\sum k^{-1}p^{-3}+\sum k^{-4}\right\}, \tag{2.8}$$

whereas for the Lorentz gauge, after including the transformation of eq. (2.2) (i.e. adding $-g_0^2(d-1)^2/(2dL)\sum k^{-1}p^{-3}$ to the diagrams of fig. 1), one obtains

$$V_2^{(L)}(c)=\frac{g_0^2}{L}\left\{\tfrac{3}{2}\left(\sum k^{-1}\right)^2+3\sum k^{-2}\right\}+\frac{g_0^2c^2}{L}\left\{-\frac{(d-3)}{3}\sum k^{-1}p^{-3}+\tfrac{5}{4}\sum k^{-4}\right\}, \tag{2.9}$$

The algebraic manipulation programme FORM [13] was used for some of the calculations.

We clearly observe a gauge dependence, which cannot be transformed away by field redefinitions, whereas furthermore they differ from the correct expression (which does coincide with what we have used in the past) to be derived in the next section*

$$V_2(c) = \frac{3g_0^2}{2L}\left(\sum k^{-1}\right)^2 + \frac{g_0^2 c^2}{6\pi^2 L}\sum k^{-1}. \tag{2.10}$$

The found discrepancies are of the same order of magnitude as the actual terms (e.g. $\sum k^{-4} = 0.0106075$, whereas $\sum(6\pi^2 k)^{-1} = -0.0076256$). We performed the hamiltonian analysis to be presented in the next section primarily to fix these discrepancies. The reason behind the gauge dependence is that it is not legitimate to take the "adiabatic approximation" of a time-independent background field, as will be clearly seen in the hamiltonian formulation. The second method [11] to compute the effective hamiltonian discussed in the introduction is based on calculating the euclidean transition function. There one also integrates the zero-momentum modes and the time dependence is therefore consistently included. We found this method too complicated at higher order. Despite vigorous attempts we were unable to reliably include the "non-adiabatic" correction in the background field method, the main obstacle being caused by the ghosts in covariant gauges. Nevertheless, at one loop (up to a possible field redefinition) one does obtain the correct result, which is why we still use the background field technique to include the lattice artifacts. The lattice calculation would otherwise be too voluminous.

3. Hamiltonian perturbation theory

Here we will describe the computation to $O(g_0^4)$ of the effective hamiltonian for the zero-momentum gauge fields, using hamiltonian perturbation theory [14]. We will not keep track of field-independent terms proportional to g_0^4, since they do not contribute to the mass spectrum. Including them would furthermore require a three-loop computation.

The SU(2) Yang–Mills hamiltonian in the Coulomb gauge $\partial_i A_i = 0$ is given by [5, 14]

$$H = \tfrac{1}{2}g_0^2\int_0^L \mathrm{d}^d x\,\mathrm{d}^d y\,\rho^{-1/2}\pi_i^a(x)\rho^{1/2}K(x,y)_{ij}^{ab}\rho^{1/2}\pi_j^b(y)\rho^{-1/2} + \frac{1}{4g_0^2}\int_0^L \mathrm{d}^d x\,F_{ij}^a(x)^2, \tag{3.1}$$

where the gauge fields are transverse ($\partial_j A_j(x) = 0$), $\pi_j(x) = E_j(x) - \partial_j\partial_k^{-2}\partial_i E_i(x)$ is

* Note that $\lim_{d\to 3}(d-3)\sum p^{-3} = -1/(2\pi^2)$ (see ref. [5]).

the transverse electric field, and ρ is the jacobian associated to the gauge fixing*

$$\rho(A) = \det'\left(-\partial_j D_j(A)\right) = \det'\left(-\partial_j^2 - i\partial_j \operatorname{ad}(A_j)\right). \tag{3.2}$$

The prime denotes that the zero modes, associated with the global gauge invariance, are omitted. The non-abelian Coulomb Green function is

$$K(x,y)_{ij} = \delta_{ij}\,\delta_3(x-y) - \operatorname{ad} A_i\,(\partial_l D_l)^{-1}\partial_k^2(\partial_m D_m)^{-1}\operatorname{ad} A_j. \tag{3.3}$$

In lowest order the hamiltonian is independent of the zero-momentum gauge fields, leading to an infinite degeneracy, which will be lifted in the next order. Bloch's perturbation theory [10] yields an effective hamiltonian in the zero-momentum gauge fields, which will describe the level splitting of the ground state of H_0 (the lowest order part of H). Writing $H = H_0 + H_1$ one can derive the following effective hamiltonian, which we computed using the algebraic programme FORM [13] from Bloch's explicit formula [5, 10] (for more details we refer to Lüscher's paper). Up to the order we will need we have

$$\begin{aligned}
&H' = \sum_i H_i', \qquad H_1' = U_1, \qquad H_2' = 0, \qquad H_3' = -\tfrac{1}{2}\{U_1, U_2\},\\
&H_4' = \tfrac{1}{2}\{U_1^2, U_3\}, \qquad H_5' = -\tfrac{1}{2}\{U_1^3, U_4\} + \tfrac{1}{2}\{U_1^2, U_{2,2}\} + \tfrac{3}{8}\{U_1, U_2^2\} + \tfrac{1}{2}U_2U_1U_2,\\
&H_6' = \tfrac{1}{2}\{U_1^4, U_5\} - \tfrac{1}{2}\{U_1^3, U_{2,3}\} - 5\{U_1^2, \{U_2, U_3\}\} + 2\{U_2, \{U_3, U_1^2\}\} + 5U_1\{U_2, U_3\}U_1\\
&\qquad + \{U_3, [U_1, [U_1, U_2]]\} - 2\{[U_1, U_2], [U_3, U_1]\} - 3\{\{U_1, U_2\}, \{U_3, U_1\}\},
\end{aligned} \tag{3.4}$$

where U is defined by

$$\begin{aligned}
U_1 &= \langle 0|\mathscr{R}|0\rangle,\\
U_{n_1, n_2, \ldots, n_p} &= \sum_{\{n_i\}} \langle 0|\mathscr{R}S^{n_{i_1}}\mathscr{R}S^{n_{i_2}}\ldots\mathscr{R}S^{n_{i_p}}\mathscr{R}|0\rangle, \qquad (n_i > 1),\\
\mathscr{R} &= H_1(1 - SH_1)^{-1} = H_1 + H_1SH_1 + H_1SH_1SH_1 + \ldots,\\
S &= \frac{1 - |0\rangle\langle 0|}{E_0 - H_0}.
\end{aligned} \tag{3.5}$$

The sum in this equation is over all different orderings of the integers $n_i > 1$ and $|0\rangle$ is the vacuum state for H_0. We now expand H_1 in g_0 and c (which is $O(g_0^{2/3})$)

* $(\operatorname{ad} A_j)^{ab} = -i\varepsilon_{abc}A_j^c$.

by substituting

$$A_k^a(\boldsymbol{x}) = c_k^a/L + g_0 q_k^a(\boldsymbol{x}) L^{(d+1)/2}, \qquad P q_k^a = 0,$$

$$\pi_k^a(\boldsymbol{x}) = e_k^a/L^2 + g_0 p_k^a(\boldsymbol{x}) L^{(d-1)/2}, \qquad P p_k^a = 0. \tag{3.6}$$

The operators e and p are the momenta conjugate to the coordinates c and q and for convenience we have chosen the scaling conventions (remember that in d dimensions g_0^2 has the dimension of L^{d-3}) such that c, e and the Fourier components $q_i(\boldsymbol{k})$ and $p_i(\boldsymbol{k})$ are dimensionless*

$$q_i^a(\boldsymbol{x}) = \frac{1}{L^d} \sum_{\boldsymbol{k} \neq \boldsymbol{0}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}/L} q_i^a(\boldsymbol{k}), \qquad p_i^a(\boldsymbol{x}) = \frac{1}{L^d} \sum_{\boldsymbol{k} \neq \boldsymbol{0}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}/L} p_i^a(\boldsymbol{k}). \tag{3.7}$$

The canonical commutation relations imply

$$\left[c_i^a, e_j^b\right] = i\delta_{ij}\delta_{ab}, \qquad e_j^b = \frac{1}{i}\frac{\partial}{\partial c_j^b},$$

$$\left[q_i^a(\boldsymbol{k}), p_j^b(\boldsymbol{l})\right] = i\delta_{ab}\delta_{\boldsymbol{k}+\boldsymbol{l},\boldsymbol{0}}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right). \tag{3.8}$$

In terms of (transverse) creation and annihilation operators

$$q_i^a(\boldsymbol{k}) = b_i^a(\boldsymbol{k}) + b_i^a(-\boldsymbol{k})^\dagger, \qquad p_i^a(\boldsymbol{k}) = -ik\left(b_i^a(\boldsymbol{k}) - b_i^a(-\boldsymbol{k})^\dagger\right),$$

$$\left[b_i^a(\boldsymbol{k}), b_j^c(\boldsymbol{k})^\dagger\right] = \frac{1}{2k}\delta_{ab}\delta_{\boldsymbol{k},\boldsymbol{l}}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right), \tag{3.9}$$

the lowest-order hamiltonian is given by

$$H_0 = \frac{1}{L}\sum_{\boldsymbol{k}\neq\boldsymbol{0}} k\left\{b_k^a(\boldsymbol{k})^\dagger, b_j^a(\boldsymbol{k})\right\} = E_0 + \frac{1}{L}\sum_{\boldsymbol{k}\neq\boldsymbol{0}} 2k b_j^a(\boldsymbol{k})^\dagger b_j^a(\boldsymbol{k}). \tag{3.10}$$

As usual, the vacuum $|0\rangle$ is defined by $b_i^a(\boldsymbol{k})|0\rangle = 0$ and one can use the creation operators to form a Fock space. We can now write H_1 in terms of the operators $b_i^a(\boldsymbol{k})$, $b_i^a(\boldsymbol{k})^\dagger$, c_i^a and e_i^a and work out the Fock space expectation values, which can be readily implemented in an algebraic programme as follows. Commute every creation operator $b_i^a(\boldsymbol{k})^\dagger$ to the left and every annihilation operator $b_i^a(\boldsymbol{k})$ to the right, using eq. (3.9) and

$$b_i^a(\boldsymbol{k}) S^n(\Delta E) = S^n(\Delta E) b_i^a(\boldsymbol{k}) + S^n(\Delta E + k),$$

$$S^n(\Delta E) b_i^a(\boldsymbol{k})^\dagger = b_i^a(\boldsymbol{k})^\dagger S^n(\Delta E) + S^n(\Delta E + k), \tag{3.11}$$

* Scaling $c \to g_0^{2/3} L^{1-d/3} c$, $q \to L^{(d-1)/2} q$ and $p \to L^{(d-1)/2} p$ will give the conventions of ref. [5].

where

$$S^n(\Delta E) = \frac{(1 - |0\rangle\langle 0|)}{(E_0 - H_0 - \Delta E)^n} . \tag{3.12}$$

After this procedure has been completed only terms with no creation or annihilation operators will survive, and they correspond as usual to the sum over all possible contractions. Furthermore, if $\Delta E \neq 0$, $S^n(\Delta E)$ is to be replaced by $(-\Delta E)^{-n}$. If $\Delta E = 0$ the projection $(1 - |0\rangle\langle 0|)$ is operative and $S^n(0)$ is to be replaced by 0.

One major complication over normal hamiltonian perturbation theory is that H_1 contains non-commuting operators c and e. Thus $U_{\{n_i\}}$ will become a hermitian operator $U_{\{n_i\}}(c, e)$ acting on the Hilbert space of wavefunctions $\psi(c)$, normalized with the standard L^2 norm on $\mathbf{R}^9$ (being the configuration space of the zero-momentum gauge fields c). If these operators would have been commuting one can show that $H_1 = U_1^c$, where U_1^c is the sum of all connected diagrams occurring in the diagrammatic expansion of U_1. It is, however, crucial that for the case at hand disconnected diagrams are taken into consideration too, because the disconnected pieces are now operators which do not commute. These additional terms are precisely of the form encountered in the previous section regarding the gauge dependence of the effective potential. Taking a time-independent background field in the lagrangian formulation is more or less equivalent to putting the commutator of e with c equal to zero in the hamiltonian formulation.

We now expand H_1 in terms of the background field. The most difficult part is the contribution due to the Coulomb Green function, which we will work out in the momentum representation. For ease of computation we will put $L = 1$. The proper L dependence can be easily recovered on the basis of dimensional arguments. In the following also momentum sums will be implicitly assumed:

$$\begin{aligned}
&\tfrac{1}{2} g_0^2 \rho^{-1/2} \pi_i^a(-l) \rho^{1/2} K(l, k)_{ij}^{ab} \rho^{1/2} \pi_j^b(k) \rho^{-1/2} \\
&\quad = \frac{g_0^2}{2} e_i^a e_i^a + \frac{g_0^2}{4} [e_i^a, [\hat{\rho}, e_i^a]] + \frac{g_0^2}{8} [e_i^a, \hat{\rho}][\hat{\rho}, e_i^a] + \tfrac{1}{2} p_i^a(-k) A_{ij}^{ab}(k) p_j^b(k) \\
&\qquad + \tfrac{1}{4} A_{ij}^{ab}(k) \big[p_i^a(-k), \big[\hat{\rho}, p_j^b(k)\big]\big] + \tfrac{1}{8} [p_i^a(-k), \hat{\rho}] A_{ij}^{ab}(k) \big[\hat{\rho}, p_j^b(k)\big] \\
&\qquad - \frac{g_0^2}{2k^2} (e_i^a + \tfrac{1}{2}[e_i^a, \hat{\rho}]) q_i^b(-k) \varepsilon_{abc} (1 + k \cdot \mathrm{ad}\, c/k^2)_{cd}^{-2} \varepsilon_{dgf} c_j^g \big(p_j^f(k) + \tfrac{1}{2}\big[\hat{\rho}, p_j^f(k)\big]\big) \\
&\qquad - \frac{g_0^2}{2k^2} (p_i^a(-k) + \tfrac{1}{2}[p_i^a(-k), \hat{\rho}]) c_i^b \varepsilon_{abc} (1 + k \cdot \mathrm{ad}\, c/k^2)_{cd}^{-2} \varepsilon_{dgf} q_j^g(k) \big(e_j^f + \tfrac{1}{2}\big[\hat{\rho}, e_j^f\big]\big) \\
&\qquad - \frac{g_0^2}{2k^2} (e_i^a + \tfrac{1}{2}[e_i^a, \hat{\rho}]) q_i^b(-k) \varepsilon_{abc} (1 + k \cdot \mathrm{ad}\, c/k^2)_{cd}^{-2} \varepsilon_{dgf} q_j^g(k) \big(e_j^f + \tfrac{1}{2}\big[\hat{\rho}, e_j^f\big]\big) \\
&\qquad + \tfrac{1}{2} (p_i^a(-l) + \tfrac{1}{2}[p_i^a(-l), \hat{\rho}])\, \Delta K_{ij}^{ab}(l, k) \big(p_j^b(k) + \tfrac{1}{2}\big[\hat{\rho}, p_j^b(k)\big]\big),
\end{aligned} \tag{3.13}$$

where

$$\hat{\rho} = \ln(\rho(A)/\rho(0)) = \operatorname{Tr}\ln\left(1 + i\,\mathrm{ad}\,A_j(x)\partial_j\partial_k^{-2}\right) \tag{3.14}$$

and ΔK is defined through

$$K_{ij}^{ab}(l,k) = A_{ij}^{ab}(k) + \Delta K_{ij}^{ab}(l,k), \tag{3.15}$$

with

$$A_{ij}(k) = \delta_{ij} + k^{-2}\,\mathrm{ad}\,c_i\left(1 + k\cdot\mathrm{ad}\,c/k^2\right)^{-2}\mathrm{ad}\,c_j. \tag{3.16}$$

Thus, ΔK is a function of q and c that vanishes for $q=0$. It contributes to n-point vertices ($n \geqslant 3$) and it is sufficient to expand ΔK to second order in c and g_0, when we want to determine the effective hamiltonian to fourth order in g_0. To complete the expansion of the hamiltonian we have (ignoring L dependence)

$$\begin{aligned}\frac{1}{4g_0^2}\int_0^L \mathrm{d}^d x\, F_{ij}^a(x)^2 = {} & \frac{1}{4g_0^2}\left(\varepsilon_{abd}c_i^a c_j^b\right)^2 + \tfrac{1}{2}q_i^a(-k)B_{ij}^{ab}(k)q_j^b(k) \\ & - ig_0 k_i q_j^a(k) q_i^b(l) q_j^c(-k-l)\varepsilon_{abc} \\ & + g_0 c_j^a q_i^b(l)\varepsilon_{abc} q_j^d(-k-l) q_i^e(k)\varepsilon_{cde} \\ & + \frac{g_0^2}{4} q_i^a(-l) q_j^b(l-m)\varepsilon_{abc} q_i^d(m-k) q_j^e(k)\varepsilon_{cde},\end{aligned} \tag{3.17}$$

where

$$B_{ij}(k) = \delta_{ij}\left(k^2 + 2k\cdot\mathrm{ad}\,c + (\mathrm{ad}\,c)^2\right) + \mathrm{ad}\,c_i\,\mathrm{ad}\,c_j - 2\,\mathrm{ad}\,c_j\,\mathrm{ad}\,c_i. \tag{3.18}$$

We will now analyse H_1 in more detail. First we consider the terms where $\hat{\rho}$ contributes. To the required order we find

$$\begin{aligned}\hat{\rho} &= -\frac{l\cdot A^a(k-l)\,k\cdot A^a(l-k)}{k^2 l^2} - \frac{k\cdot A^a(l-k)\,l\cdot A^a(m-l)\,m\cdot A^b(n-m)\,n\cdot A^b(m-n)}{2k^2l^2m^2n^2} \\ &= -g_0^2\frac{l\cdot q^a(k-l)\,k\cdot q^b(l-k)}{k^2l^2}\left\{\delta_{ab} + \frac{(k\cdot c^d)^2\delta_{ab} + (k\cdot c^a)(k\cdot c^b)}{k^4} - \frac{(k\cdot c^a)(l\cdot c^b)}{k^2l^2}\right\} \\ &\quad - \frac{(k\cdot c^a)^2}{k^4} - \frac{\left[(k\cdot c^a)^2\right]^2}{2k^8}.\end{aligned} \tag{3.19}$$

This allows us to evaluate (ignoring a field-independent $O(g_0^4)$ term)

$$h_0^{(1)} = \tfrac{1}{4}A_{ij}^{ab}(\boldsymbol{k})\left[p_i^a(-\boldsymbol{k}),\left[\hat{\rho},p_j^b(\boldsymbol{k})\right]\right] + \tfrac{1}{8}\left[p_i^a(-\boldsymbol{k}),\hat{\rho}\right]A_{ij}^{ab}(\boldsymbol{k})\left[\hat{\rho},p_j^b(\boldsymbol{k})\right]$$

$$+\frac{g_0^2}{4}[e_i^a,[\hat{\rho},e_i^a]] + \frac{g_0^2}{8}[e_i^a,\hat{\rho}][\hat{\rho},e_i^a]$$

$$= \sum_{\boldsymbol{k}+\boldsymbol{l}+\boldsymbol{m}=\boldsymbol{0}} \left\{ \frac{3g_0^2\left((\boldsymbol{k}\cdot\boldsymbol{l})^2 - k^2l^2\right)}{2k^2l^2m^2} + \frac{g_0^2c^2\left(k^2l^2-(\boldsymbol{k}\cdot\boldsymbol{l})^2\right)}{4dk^2l^2m^2}\left(\frac{l^2}{k^2m^2} - \frac{14}{m^2}\right)\right\}$$

$$-\frac{9g_0^2}{2dk^2} - \frac{15g_0^2c^2}{2d^2k^4} + \frac{g_0^2c^2}{2d^2k^2l^2}\,. \tag{3.20}$$

One can use for example that $\sum_{\boldsymbol{k}\neq\boldsymbol{0}}(\boldsymbol{k}\cdot\boldsymbol{c}^a)^2k^{-6} = \sum k_1^2c^2k^{-6} = \sum c^2d^{-1}k^{-4}$. Alternatively, for any analytic function $F(c)$ invariant under the cubic symmetry

$$F(c) = F(0) + \frac{c^2}{18}\sum_{a,i=1}^{3}\frac{\partial^2F}{\partial c_i^{a2}}(0) + O(c^4)\,. \tag{3.21}$$

This identity will also play an important role in the two-loop calculation.

The only other terms where $\hat{\rho}$ can contribute to the effective hamiltonian to $O(g_0^4)$ are

$$h_0^{(2)} = -\frac{g_0^2}{4k^2}[e_i^a,\hat{\rho}]q_i^b(-\boldsymbol{k})\varepsilon_{abd}\varepsilon_{dfg}c_j^fp_j^g(\boldsymbol{k}) - \frac{g_0^2}{4k^2}p_i^a(-\boldsymbol{k})c_i^b\varepsilon_{abd}\varepsilon_{dfg}q_j^f(\boldsymbol{k})[e_j^g,\hat{\rho}]$$

$$= \frac{g_0^2c^2(d-1)}{d^2k^2l^2}\,. \tag{3.22}$$

The hamiltonian truncated to the zero-momentum fields contained in H_1 is given by

$$h_1 = \frac{g_0^2}{2}e_i^ae_i^a + \frac{1}{4g_0^2}\left(\varepsilon_{abd}c_i^ac_j^b\right)^2. \tag{3.23}$$

Next we consider the terms in H_1 that are quadratic in p and q. Those that contain e are given by

$$h_2 = -\frac{g_0^2}{2k^2} e_i^a q_i^b(-\boldsymbol{k}) \varepsilon_{abc} (1 + \boldsymbol{k} \cdot \mathrm{ad}\, \boldsymbol{c}/k^2)_{cd}^{-2} \varepsilon_{dgf} c_j^g p_j^f(\boldsymbol{k})$$

$$- \frac{g_0^2}{2k^2} p_i^a(-\boldsymbol{k}) c_i^b \varepsilon_{abc} (1 + \boldsymbol{k} \cdot \mathrm{ad}\, \boldsymbol{c}/k^2)_{cd}^{-2} \varepsilon_{dgf} q_j^g(\boldsymbol{k}) e_j^f$$

$$- \frac{g_0^2}{2k^2} e_i^a q_i^b(-\boldsymbol{k}) \varepsilon_{abc} (1 + \boldsymbol{k} \cdot \mathrm{ad}\, \boldsymbol{c}/k^2)_{cd}^{-2} \varepsilon_{dgf} q_j^g(\boldsymbol{k}) e_j^f . \qquad (3.24)$$

The term not containing e is

$$h_3 = \tfrac{1}{2} p_i^a(-\boldsymbol{k}) A_{ij}^{ab}(\boldsymbol{k}) p_j^b(\boldsymbol{k}) + \tfrac{1}{2} q_i^a(-\boldsymbol{k}) B_{ij}^{ab}(\boldsymbol{k}) q_j^b(\boldsymbol{k}) . \qquad (3.25)$$

Finally we have to consider in H_1 the three- and four-point vertices, which can be somewhat simplified if we take into consideration that they only contribute at the two-loop level and therefore give at least an $O(g_0^2)$ contribution to the effective hamiltonian (or any of the operators U). Furthermore up to $O(g_0^4)$ they can only mix with h_1 and h_3 (h_2 can at most contribute to the $O(g_0^4)$-field-independent constant, which is not included). Since h_1 does not depend on p and q, it can only contribute through disconnected diagrams and therefore enters through commutator terms (since if h_1 would commute with everything, the disconnected diagrams would precisely cancel). Again this implies that this will at most contribute to the $O(g_0^4)$-field-independent term. We are left with mixing with h_3 only. Therefore, at two-loop e does not enter in the terms in H_1 that contribute to the effective hamiltonian to $O(g_0^4)$, which implies that we only need to consider connected two-loop diagrams. Any term in H_1 that is proportional to c^2, will contribute at two loop to the effective hamiltonian proportional to $g_0^2 c^2$. This implies we can perform the "projection" of eq. (3.21) to the $O(c^2)$ terms in H_1. This will lead to some considerable simplification. Collecting the "projected" three-point vertices yields

$$h_4 = -i g_0 k_i q_j^a(\boldsymbol{k}) q_i^b(\boldsymbol{l}) q_j^c(-\boldsymbol{k}-\boldsymbol{l}) \varepsilon_{abc} + g_0 c_j^a q_i^b(\boldsymbol{l}) \varepsilon_{abc} q_j^d(-\boldsymbol{k}-\boldsymbol{l}) q_i^e(\boldsymbol{k}) \varepsilon_{cde}$$

$$- \frac{g_0}{k^2} p_i^a(\boldsymbol{l}) q_i^b(-\boldsymbol{k}-\boldsymbol{l}) \varepsilon_{abc} c_j^d p_j^e(\boldsymbol{k}) \varepsilon_{cde} - i \frac{g_0 c^2}{3dk^2 l^2} p_i^a(\boldsymbol{l}) k_j q_j^b(-\boldsymbol{k}-\boldsymbol{l}) p_i^c(\boldsymbol{k}) \varepsilon_{abc} ,$$

$$(3.26)$$

where the last term underwent simplification, whereas other terms vanished due to transversality (after having applied the "projection"). Similar manipulations will reduce the number of terms in the four-point vertex

$$h_5 = \frac{g_0^2}{4} q_i^a(-l) q_j^b(l-m) \varepsilon_{abc} q_i^d(m-k) q_j^e(k) \varepsilon_{cde}$$

$$- i \frac{2g_0^2}{m^2k^2} p_i^a(-l) q_i^b(l-m) \varepsilon_{abc} \varepsilon_{cde} m \cdot q^e(m-k) \varepsilon_{dfg} c_j^f p_j^g(k)$$

$$- \frac{g_0^2}{2m^2} p_i^a(-l) q_i^b(l-m) \varepsilon_{abc} q_j^d(m-k) p_j^e(k) \varepsilon_{def} \left(\delta_{cf} + 2i\varepsilon_{cfg} \frac{m \cdot c^g}{m^2} + \delta_{cf} \frac{2c^2}{dm^2} \right)$$

$$+ \frac{g_0^2 c^2}{2dm^2 l^2} \left\{ \frac{1}{k^2} [p_i^a(-l) l \cdot q^b(l-m) m \cdot q^d(m-k) p_i^e(k)] (\delta_{ab}\delta_{de} + \delta_{ae}\delta_{bd}) \right.$$

$$\left. + \frac{2}{m^2} [m \cdot p^a(-l) l \cdot q^b(l-m) q_i^d(m-k) p_i^e(k)] (\delta_{ae}\delta_{bd} - \delta_{ad}\delta_{be}) \right\}. \quad (3.27)$$

We can simplify the two-loop calculation even further, by diagonalizing the quadratic part of the hamiltonian for an abelian background field, which can then take over the role of H_0 (the same technique was applied in eqs. (2.25) and (2.26) for the path-integral formulation in ref. [6]). We need to achieve this only to second order in c, which is indeed determined by an abelian background field through the use of gauge invariance. First we apply to this order the canonical transformation

$$q_i(k) = \left(\delta_{ij} + \frac{1}{2k^2} \operatorname{ad} c_i \operatorname{ad} c_j \right) \hat{q}_j(k) \qquad \left[= \left(1 + \frac{c^2}{3dk^2} \right) \hat{q}_i(k) \right],$$

$$p_i(k) = \left(\delta_{ij} - \frac{1}{2k^2} \operatorname{ad} c_i \operatorname{ad} c_j \right) \hat{p}_j(k) \qquad \left[= \left(1 - \frac{c^2}{3dk^2} \right) \hat{p}_i(k) \right]. \quad (3.28)$$

The terms between the square brackets follow after the "projection", eq. (3.21). In terms of the new coordinates and momenta the quadratic part of the hamiltonian is given by

$$\hat{H}_0 = \sum_{k \neq 0} \tfrac{1}{2} \hat{p}_i^a(-k) \hat{p}_i^a(k) + \tfrac{1}{2} \hat{q}_i^a(-k) (k + \operatorname{ad} c)_{ab}^2 \hat{q}_i^b(k). \quad (3.29)$$

Let us take $c_i^a = c_i \delta_{a3}$ and define

$$\hat{q}_i^{\pm}(\boldsymbol{k}) = \frac{\sqrt{2}}{2}\left(\hat{q}_i^1(\boldsymbol{k}) \mp \hat{q}_i^2(\boldsymbol{k})\right), \qquad \hat{q}_i^0(\boldsymbol{k}) = \hat{q}_i^3(\boldsymbol{k}),$$

$$\hat{p}_i^{\pm}(\boldsymbol{k}) = \frac{\sqrt{2}}{2}\left(\hat{p}_i^1(\boldsymbol{k}) \mp \hat{p}_i^2(\boldsymbol{k})\right), \qquad \hat{p}_i^0(\boldsymbol{k}) = \hat{p}_i^3(\boldsymbol{k}), \tag{3.30}$$

in terms of which

$$\hat{H}_0 = \sum_{\alpha \in \{0, \pm\}, \boldsymbol{k} \neq \boldsymbol{0}} \tfrac{1}{2}\hat{p}_i^{-\alpha}(-\boldsymbol{k})\hat{p}_i^{\alpha}(\boldsymbol{k}) + \tfrac{1}{2}(\boldsymbol{k} + \alpha \boldsymbol{c})^2 \hat{q}_i^{-\alpha}(-\boldsymbol{k})\hat{q}_i^{\alpha}(\boldsymbol{k}). \tag{3.31}$$

If $\hat{h}_4$ and $\hat{h}_5$ are the three- and four-point functions expressed in terms of the transformed momenta and coordinates and if $|\hat{0}\rangle$ is the (c-dependent) vacuum for $\hat{H}_0$ (i.e. $\hat{H}_0|\hat{0}\rangle = \hat{E}_0|\hat{0}\rangle = \frac{1}{2}\sum_{\alpha, \boldsymbol{k} \neq \boldsymbol{0}} |\boldsymbol{k} + \alpha \boldsymbol{c}|\, |\hat{0}\rangle$), then the two-loop contribution to the effective hamiltonian H' takes the form

$$\langle \hat{0}|\hat{h}_4 \frac{(1 - |\hat{0}\rangle\langle\hat{0}|)}{(\hat{E}_0 - \hat{H}_0)} \hat{h}_4|\hat{0}\rangle + \langle\hat{0}|\hat{h}_5|\hat{0}\rangle, \tag{3.32}$$

which can be computed most easily using Wick's theorem and

$$\left\langle \hat{0}|\hat{q}_i^{\alpha}(\boldsymbol{k})\hat{q}_j^{\beta}(\boldsymbol{l})|\hat{0}\right\rangle = \frac{\delta_{\alpha+\beta,0}\delta_{\boldsymbol{k}+\boldsymbol{l},\boldsymbol{0}}}{2|\boldsymbol{k} + \alpha \boldsymbol{c}|}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right),$$

$$\left\langle \hat{0}|\hat{p}_i^{\alpha}(\boldsymbol{k})\hat{p}_j^{\beta}(\boldsymbol{l})|\hat{0}\right\rangle = \tfrac{1}{2}|\boldsymbol{k} + \alpha \boldsymbol{c}|\delta_{\alpha+\beta,0}\delta_{\boldsymbol{k}+\boldsymbol{l},\boldsymbol{0}}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right),$$

$$\left\langle \hat{0}|\hat{p}_i^{\alpha}(\boldsymbol{k})\hat{q}_j^{\beta}(\boldsymbol{l})|\hat{0}\right\rangle = -\frac{i}{2}\delta_{\alpha+\beta,0}\delta_{\boldsymbol{k}+\boldsymbol{l},\boldsymbol{0}}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right),$$

$$\left\langle \hat{0}|\hat{q}_i^{\alpha}(\boldsymbol{k})\hat{p}_j^{\beta}(\boldsymbol{l})|\hat{0}\right\rangle = +\frac{i}{2}\delta_{\alpha+\beta,0}\delta_{\boldsymbol{k}+\boldsymbol{l},\boldsymbol{0}}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right). \tag{3.33}$$

Still a considerable amount of algebra remains due to the complexity of the three-and four-point vertices. We found the following result

$$\langle\hat{0}|\hat{h}_4\frac{(1-|\hat{0}\rangle\langle\hat{0}|)}{(\hat{E}_0-\hat{H}_0)}\hat{h}_4|\hat{0}\rangle$$

$$=g_0^2\sum_{k+l+m=0}\Bigg\{-\frac{c^2(d-1)(3l^2+(k-m)^2)((d-2)k^2m^2+(k\cdot m)^2)}{4dk^3l^3m^3(k+l+m)}$$

$$-\frac{c^2(k^2+l^2+m^2-2kl-2km-2lm)}{12dk^2l^2m^2}\left(d-3+3\frac{(k\cdot l)^2}{k^2l^2}-\frac{(k\cdot l)(l\cdot m)(m\cdot k)}{k^2l^2m^2}\right)$$

$$+\frac{(k^2m^2-(k\cdot m)^2)((d-2)k^2m^2+(k\cdot m)^2-l^2(k\cdot m)^2)}{k^2l^2m^2}\Bigg[-\frac{3}{2klm(k+l+m)}$$

$$+\frac{c^2(d-1)(k^{-1}+l^{-1}+m^{-1})}{2dklm(k+l+m)^2}+\frac{c^2(d-6)(k^{-2}+l^{-2}+m^{-2})}{2dklm(k+l+m)}$$

$$+\frac{c^2(m^3k-m^3l+l^3k+2(m\cdot l)^2+4m^2l^2-2kml^2-2m^2lk)}{2dk^3l^3m^3(k+l+m)}\Bigg]\Bigg\}, \tag{3.34}$$

$$\langle\hat{0}|\hat{h}_5|\hat{0}\rangle=g_0^2\sum_{k+l+m=0}\Bigg\{\frac{3c^2\left(m^2l^2-(m\cdot l)^2\right)}{2dk^2l^2m^2}\Bigg[\frac{2(k\cdot l)}{3l^3k}+\frac{(d-1)}{kl}-\frac{(k\cdot l)}{2k^2l^2}$$

$$-\left(\frac{k}{l}-1\right)\frac{(k\cdot l)}{k^2m^2}\Bigg]+\frac{\left((d-2)k^2l^2+(k\cdot l)^2\right)}{k^2l^2m^2}\Bigg(\frac{c^2}{2d}\Bigg[\frac{1}{2m^2}\left(\frac{5k}{l}+\frac{k^3}{l^3}-6\right)$$

$$-\frac{(d-2)}{2kl}\left(\frac{l^2}{k^2}-1\right)+\frac{l}{k^3}+\frac{(k\cdot l)}{2k^3l}\Bigg]+\frac{3}{4}\Bigg[\frac{l}{k}-1\Bigg]\Bigg)\Bigg\}$$

$$+g_0^2\sum_{k\neq 0,\,l\neq 0}\left(\frac{(d^2-3d+3)k^2l^2-(k\cdot l)^2}{8k^3l^3}\right)\left(3-\frac{2c^2(d-5)}{dk^2}\right). \tag{3.35}$$

For the same reasons as given for the two-loop contributions, $h_0^{(1)}$ and $h_0^{(2)}$ do not mix with the other pieces in H_1 to $O(g_0^4)$. Adding together what we have so far

$$H'_{(1)} = h_0^{(1)} + h_0^{(2)} + \langle \hat{0} | \hat{h}_4 \frac{(1 - |\hat{0}\rangle\langle\hat{0}|)}{(\hat{E}_0 - \hat{H}_0)} \hat{h}_4 | \hat{0}\rangle + \langle \hat{0} | \hat{h}_5 | \hat{0}\rangle$$

$$= \frac{3g_0^2}{2kl} - \frac{(d-3)g_0^2c^2}{3k^3l} + \left[\frac{9g_0^2}{4k^2} + \frac{g_0^2c^2}{4k^4} + \frac{8g_0^2c^2}{9k^3l} + \frac{14(d-3)g_0^2c^2}{27k^3l}\right] + O(d-3). \tag{3.36}$$

The necessity of quite a few "miraculous" cancellations leaves no room for errors in the final answer. Some of the terms are placed between square brackets in anticipation of more magic to come.

All that remains to be done is a one-loop calculation of the operators $U_{\{n_i\}}$ with $H_1 = h_1 + h_2 + h_3$ and combine those to obtain the one-loop contribution (including disconnected parts) to the effective hamiltonian. We discussed earlier how this can be performed by algebraic manipulation, which we have used as an independent check on our calculations by hand. If we retain in H_1 only h_3, the contribution to the effective hamiltonian corresponds to the groundstate energy of a harmonic c-dependent potential. This truncation is what in the lagrangian formulation corresponds to restricting the background field to be time-independent. We have applied the techniques of appendix E of ref. [5] to compute this truncated one-loop result to $O(c^6)$. The result ($H'_{(2)}$) is again of the form of eq. (2.1) and the relevant coefficients are given in table 1, for comparison with the background field results in the Lorentz and non-local gauges. Also included is α_1, which is obtained by applying eq. (3.4) with $H_1(c=0, e)$, rather than $H_1(c, e=0)$. We conclude by addressing the contributions that mix e and c, where the ordering of the operators is important. We chose to evaluate them by considering $H' = (H'_L + H'_R)/2$, where in $H'_{L(R)}$ all momentum operators e are commuted to the left (right). This is again something that is easy to implement with algebraic manipulation. Note that any term that will not depend on e is hence due to disconnected diagrams. Putting things together we find

$$H' = H'_{(1)} + H'_{(2)} + H'_{(3)} + H'_{(4)} + H'_{(5)} + \text{const.} \times g_0^4, \tag{3.37}$$

where $H'_{(3)}$ ($H'_{(4)}$) will contain the disconnected contributions to be added to the one-loop (two-loop) terms, whereas $H'_{(5)}$ gives the mixed terms in the effective hamiltonian.

$$H'_{(3)}=\left(-\frac{41}{96k^5}+\frac{21k_1^4}{16k^9}\right)F_{ij}^2r_l^2+\left(\frac{21}{32k^5}-\frac{105k_1^4}{32k^9}\right)F_{ij}^2r_j^2+\left(\frac{11}{8k^5}-\frac{63k_1^4}{16k^9}\right)\det{}^2 c, \tag{3.38}$$

$$H'_{(4)}=-\frac{9g_0^2}{4k^2}-\frac{g_0^2}{4k^4}-\frac{g_0^2c^2(d-1)^3}{8d^2k^3l}, \tag{3.39}$$

$$\begin{aligned}H'_{(5)}=\frac{g_0^4}{2}\Big(&\beta_1\sum_{i\neq j}\{c_i^ac_i^a,e_j^be_j^b\}+\beta_2\sum_{i\neq j}\{c_i^ac_j^b,e_i^ae_j^b\}+\beta_3\sum_i\{c_i^ac_i^a,e_i^be_i^b\}\\&+\beta_4\sum_i\{c_i^ac_i^b,e_i^ae_i^b\}+\beta_5\sum_{i\neq j}\{c_i^ac_j^a,e_i^be_j^b\}+\beta_6\sum_{i\neq j}\{c_i^ac_j^b,e_j^ae_i^b\}\\&+\beta_7\sum_{i\neq j}\{c_i^ac_i^b,e_j^ae_j^b\}\Big).\end{aligned} \tag{3.40}$$

The coefficients β_i are given in table 2. The following unitary transformation

$$\Omega(c,e)=\exp\big(ig_0^2\omega(c,e)\big),\qquad \omega(c,e)=\tfrac{1}{2}\theta_1\{c_i^a,e_i^a\},\qquad \theta_1=-\frac{(5+3d)}{8dk^3} \tag{3.41}$$

will implement the necessary field renormalization. Together with the one-loop renormalization of the coupling constant

$$\frac{L^{d-3}}{g_0^2}=-\frac{11}{12\pi^2}\left(\frac{1}{(d-3)}+\ln(L\Lambda_{\mathrm{MS}})\right)=-\frac{11}{12\pi^2}\frac{1}{(d-3)}+\frac{1}{g^2(L)}, \tag{3.42}$$

where $g(L)$ is the L-dependent renormalized coupling constant, the effective hamiltonian becomes finite and of the form

$$H'_{\mathrm{R}}=\Omega H'\Omega^\dagger=\mathscr{H}_1+\mathscr{H}_2, \tag{3.43}$$

where

$$\mathscr{H}_1=\frac{g^2(L)}{2L}\left[1+g^2(L)\left(\alpha_1-\frac{11}{12\pi^2(d-3)}\right)\right]^{-1}e^2+V_1(c)+V_2(c), \tag{3.44}$$

TABLE 2

The coefficients as occurring in the effective hamiltonians, eqs. (3.47) and (3.43). The coefficients are to be summed over all non-zero momenta, where $\boldsymbol{k} \in (2\pi\mathbb{Z})^3$ and $k \equiv |\boldsymbol{k}|$. When the momentum sums are divergent in three dimensions we give the coefficients as a function of d. The coefficients γ_i were given in table 1

	$\hat{H}'_{\mathrm{R}} = \hat{\mathscr{H}}_1 + \hat{\mathscr{H}}_2$	$H'_{\mathrm{R}} = \mathscr{H}_1 + \mathscr{H}_2$
$\alpha_1(d)$	$-\frac{7d+1}{4dk^3}$	$-\frac{7d+1}{4dk^3}$
$\alpha_2(d)$	$\frac{d^2-15d+6}{4dk^3} + \frac{5(d-1)k_1^2k_2^2}{k^7}$	$\frac{d^2-15d+6}{4dk^3} + \frac{5(d-1)k_1^2k_2^2}{k^7}$
α_3	$\frac{583}{96k^5} - \frac{1029k_1^4}{16k^9} + \frac{189k_1^6}{4k^{11}}$	$\frac{93}{16k^5} - \frac{251k_1^4}{4k^9} + \frac{189k_1^6}{4k^{11}}$
α_4	$-\frac{469}{32k^5} + \frac{4865k_1^4}{32k^9} - \frac{441k_1^6}{4k^{11}}$	$-\frac{111}{8k^5} + \frac{1185k_1^4}{8k^9} - \frac{441k_1^6}{4k^{11}}$
α_5	$-\frac{39}{4k^5} + \frac{1407k_1^4}{16k^9} - \frac{63k_1^6}{k^{11}}$	$-\frac{81}{8k^5} + \frac{333k_1^4}{4k^9} - \frac{63k_1^6}{k^{11}}$
β_1	$-\frac{7}{96k^5} + \frac{27k_1^4}{32k^9}$	$\frac{1}{4k^5} + \frac{k_1^4}{16k^9}$
β_2	$-\frac{21}{32k^5} + \frac{59k_1^4}{32k^9}$	$-\frac{1}{96k^5} + \frac{9k_1^4}{32k^9}$
β_3	$\frac{23}{24k^5} - \frac{27k_1^4}{16k^9}$	$\frac{3}{8k^5} - \frac{k_1^4}{8k^9}$
β_4	$\frac{11}{6k^5} - \frac{59k_1^4}{16k^9}$	$\frac{2}{3k^5} - \frac{9k_1^4}{16k^9}$
β_5	$\frac{3}{16k^5} + \frac{27k_1^4}{16k^9}$	$\frac{1}{16k^5} + \frac{k_1^4}{8k^9}$
β_6	$\frac{1}{96k^5} + \frac{59k_1^4}{32k^9}$	$-\frac{11}{96k^5} + \frac{9k_1^4}{32k^9}$
β_7	$\frac{61}{96k^5} + \frac{59k_1^4}{32k^9}$	$\frac{49}{96k^5} + \frac{9k_1^4}{32k^9}$

with $V_1(c)$ as in eq. (2.1) and*

$$V_2(c)=\frac{1}{L}\left(H'_{(1)}+H'_{(4)}+2\gamma_1\theta_1 g_0^2c^2\right)=\frac{3g_0^2}{2L}\left(\sum k^{-1}\right)^2+\frac{g_0^2c^2}{6\pi^2L}\sum k^{-1}, \quad (3.45)$$

Whereas $\mathscr{H}_2=H'_{(5)}$ are the $O(g_0^4)$ mixed terms which were ignored in the past. The coefficients α_i, β_i and γ_i are given in table 2. As usual we assume that the bare coupling constant in V_2 and $H'_{(5)}$ can be replaced by the renormalized coupling constant.

Finally we can use the unitary transformation $\hat{\Omega}(c,e)=\exp(ig_0^2\hat{\omega}(c,e))$

$$\hat{\omega}(c,e)=\tfrac{1}{2}\left(\theta_2\{c_i^ac_i^ac_j^b,e_j^b\}+\theta_3\{c_i^ac_j^ac_i^b,e_j^b\}+\theta_4\{c_i^ac_i^ac_i^b,e_i^b\}\right),$$

$$\theta_2=\frac{31k^4-75k_1^4}{96k^9},\qquad \theta_3=-\frac{2k^4+25k_1^4}{16k^9},\qquad \theta_3=-\frac{25k^4-125k_1^4}{32k^9}, \quad (3.46)$$

to transform the effective hamiltonian H'_R to

$$\hat{H}'_R=\hat{\Omega}H'_R\hat{\Omega}^\dagger=\hat{\mathscr{H}}_1+\hat{\mathscr{H}}_2, \quad (3.47)$$

where $\hat{\mathscr{H}}_1$ is up to $O(g_0^4)$ precisely the effective hamiltonian we have used in the past [6], whereas $\hat{\mathscr{H}}_2$ is now given by

$$\hat{\mathscr{H}}_2=H'_{(5)}+i\frac{g_0^4}{2L}[\hat{\omega},e^2]. \quad (3.48)$$

The operators $\hat{\mathscr{H}}_1$ and $\hat{\mathscr{H}}_2$ are of the same form as $\mathscr{H}_1$ and $\mathscr{H}_2$, but with different coefficients α_i and β_i, which are also given in table 2. Note that the coefficients α_i now correspond to those of the background field calculation in the non-local gauge (since this was the reason for introducing $\hat{\Omega}$). It should be noted that the coefficients α_3, α_4 and α_5 can be entirely transformed away by a unitary transformation similar to $\hat{\Omega}$, however, at the expense of changing $\mathscr{H}_2$. Conversely, $\mathscr{H}_2$ *cannot* be completely transformed away. One should thus keep in mind that it is not a priori obvious that $\hat{\mathscr{H}}_2$ will have a negligible influence on the spectrum. It is outside the scope of this paper to investigate this in further detail. What we did investigate though, is that taking the coefficients belonging to $\mathscr{H}_1$, rather than $\hat{\mathscr{H}}_1$, changed the results for the mass ratios by at most 1%.

It is reassuring that the intuition expressed in ref. [12] concerning the two-loop effective potential (by the statement that all "non-local" terms, i.e. terms depending only on a single momentum sum, should cancel) is confirmed to the order we

* The third term comes form $ig_0^2\gamma_1[\omega,c^2]$. We thus see that the terms between square brackets in eq. (3.36) for $H'_{(1)}$ exactly cancel.

investigated. We will thus assume that the two-loop effective potential restricted to abelian zero-momentum fields $c_i^a = \delta_{a3} r_i, V_1^{\mathrm{ab}}(r) = V_1(c)$ is to *all* orders still given by (see eq. (2.26), ref. [6])

$$V_2^{\mathrm{ab}}(r) = \frac{g^2 L}{64} \sum_{\alpha \neq \beta, \alpha, \beta \in \{0, \pm\}} \frac{\partial^2}{\partial r_i^2} V_1^{\mathrm{ab}}(\alpha r) \frac{\partial^2}{\partial r_j^2} V_1^{\mathrm{ab}}(\beta r). \tag{3.49}$$

4. Lattice perturbation theory at one loop

We start with Wilson's action [15] for a lattice of size $N_0 \times N_1 \times N_2 \times N_3$ with periodic boundary conditions in the limit $N_0 \to \infty$. The analysis will therefore not include finite temperature corrections.

$$S = \frac{1}{g_0^2} \sum_{\mu, \nu, x} \mathrm{Tr}\left(1 - U_{x, x+\hat{\mu}} U_{x+\hat{\mu}, x+\hat{\mu}+\hat{\nu}} U^\dagger_{x+\hat{\nu}, x+\hat{\mu}+\hat{\nu}} U^\dagger_{x, x+\hat{\nu}}\right). \tag{4.1}$$

We split the link variables into zero and non-zero-momentum contributions as follows

$$U_{x, x+\hat{\mu}} = U_\mu^{(0)}(t) U_\mu(x) = \exp\left(i c_\mu(t)/N_\mu\right) \exp\left(i g_0 q_\mu(x)\right), \tag{4.2}$$

$$P q_\mu = \frac{1}{N_1 N_2 N_3} \sum_x q_\mu(x) = 0. \tag{4.3}$$

Lattice background field calculations were first performed in ref. [16], where the analogous decomposition

$$U_{x, x+\hat{\mu}} = \hat{U}_\mu(x) U_\mu^{(0)}(t) = \exp\left(i g_0 \hat{q}_\mu(x)\right) \exp\left(i c_\mu(t)/N_\mu\right), \tag{4.4}$$

was used. The relation between the two formulations is expressed through

$$\hat{q}_\mu(x) = U_\mu^{(0)}(t)^\dagger q_\mu(x) U_\mu^{(0)}(t). \tag{4.5}$$

Note that this transformation preserves the zero-momentum projection ($P\hat{q}_\mu = 0$). We can introduce the covariant derivatives

$$D_\mu(\mathcal{O})(x) = \mathcal{O}(x + \hat{\mu}) - U_\mu^{(0)}(t)^\dagger \mathcal{O}(x) U_\mu^{(0)}(t), \tag{4.6}$$

$$\hat{D}_\mu(\mathcal{O})(x) = U_\mu^{(0)}(t) \mathcal{O}(x + \hat{\mu}) U_\mu^{(0)}(t)^\dagger - \mathcal{O}(x), \tag{4.7}$$

and their conjugates

$$D_\mu^\dagger(\mathscr{O})(x)=\mathscr{O}(x-\hat{\mu})-U_\mu^{(0)}(t)\mathscr{O}(x)U_\mu^{(0)}(t)^\dagger, \tag{4.8}$$

$$\hat{D}_\mu^\dagger(\mathscr{O})(x)=U_\mu^{(0)}(t)^\dagger\mathscr{O}(x-\hat{\mu})U_\mu^{(0)}(t)-\mathscr{O}(x). \tag{4.9}$$

The background gauge fixing is given by

$$\chi=(1-P)\sum_\mu D_\mu^\dagger(q_\mu)(x), \tag{4.10}$$

$$\hat{\chi}=(1-P)\sum_\mu \hat{D}_\mu^\dagger(\hat{q}_\mu)(x). \tag{4.11}$$

One can now work out the variation of $\hat{\chi}$ (for χ one has similar results) under an infinitesimal gauge transformation

$$U_{x,x+\hat{\mu}}\to\exp(-i\Lambda(x))U_{x,x+\hat{\mu}}\exp(i\Lambda(x+\hat{\mu})). \tag{4.12}$$

After some algebra eq. (4.12) is seen to imply [17, 18, 23]

$$\delta_\Lambda\hat{q}_\mu(x)=(1-P)W_\mu(x),\qquad\left(\delta_\Lambda U_\mu^{(0)}(t)\right)U_\mu^{(0)}(t)^\dagger=PW_\mu(x),$$

$$W_\mu(x)\equiv\left\{1-\exp(-ig_0\,\mathrm{ad}\,\hat{q}_\mu(x))\right\}^{-1}ig_0\,\mathrm{ad}\,\hat{q}_\mu(x)\,\hat{D}_\mu(\Lambda)(x)+ig_0\,\mathrm{ad}\,\hat{q}_\mu(x)(\Lambda(x)), \tag{4.13}$$

which yields for the Faddeev–Popov determinant $\det(\hat{\mathscr{M}})$

$$\hat{\mathscr{M}}=-\frac{\delta\hat{\chi}}{\delta\Lambda}=(1-P)\Bigg\{\hat{D}_\mu^\dagger\left[\left(1+\frac{i}{2}g_0\,\mathrm{ad}\,\hat{q}_\mu(x)-\frac{g_0^2}{12}\,\mathrm{ad}^2\,\hat{q}_\mu(x)\right)\hat{D}_\mu+ig_0\,\mathrm{ad}\,\hat{q}_\mu(x)\right]$$
$$+\left(\hat{D}_\mu^\dagger+1\right)ig_0\,\mathrm{ad}\,\hat{q}_\mu(x)P\left(\hat{D}_\mu+ig_0\,\mathrm{ad}\,\hat{q}_\mu(x)\left(1+\tfrac{1}{2}\hat{D}_\mu\right)\right)\Bigg\}+\mathrm{O}\left(g_0^3\hat{q}^3\right). \tag{4.14}$$

This result will be used in sect. 6, but for one-loop computations one can take as usual

$$\hat{\mathscr{M}}=\sum_\mu\hat{D}_\mu^\dagger\hat{D}_\mu,\qquad\mathscr{M}=\sum_\mu D_\mu^\dagger D_\mu. \tag{4.15}$$

It is tedious, but straightforward to expand the action in the non-zero momentum modes

$$S = S_0 + S_2 + g_0 S_3 + g_0^2 S_4 + O(g_0^3 q^5), \tag{4.16}$$

where S_0 equals the tree-level contribution to the effective action.

We introduce $(c_\mu(t+\hat{\nu}) \equiv c_\mu(t+\delta_{\nu,0}), c_0 = 0)$

$$S_{\mu\nu}(c(t)) \equiv e^{-ic_\mu(t+\hat{\nu})/N_\mu} e^{-ic_\nu(t)/N_\nu} e^{ic_\mu(t)/N_\mu} e^{ic_\nu(t+\hat{\mu})/N_\nu}, \tag{4.17}$$

and decompose $\frac{1}{2}(1 - S_{\mu\nu})$ in its hermitian and antihermitian components, $\frac{1}{2}(1 - S_{\mu\nu}) = S^+_{\mu\nu} + S^-_{\mu\nu}$, with

$$S^+_{\mu\nu} = 2 - \left(S_{\mu\nu} + S^\dagger_{\mu\nu}\right) = \mathrm{Tr}(1 - S_{\mu\nu}), \tag{4.18}$$

$$S^-_{\mu\nu} = -\left(S_{\mu\nu} - S^\dagger_{\mu\nu}\right), \tag{4.19}$$

The tree-level action S_0 is now given by

$$S_0 = \frac{N_1 N_2 N_3}{g_0^2} \sum_{\mu,\nu,t} S^+_{\mu\nu}(c(t)), \tag{4.20}$$

which takes a simple form in SU(2) when using

$$e^{ic_k/N_k} = e^{ic_k^a \sigma_a/(2N_k)} = \cos\left(\frac{r_k}{2N_k}\right) + i\frac{c_k^a \sigma_a}{r_k}\sin\left(\frac{r_k}{2N_k}\right) \tag{4.21}$$

and employing the coordinates on the unit three-sphere

$$z_k^a = \sin\left(\frac{r_k}{2N_k}\right)\frac{c_k^a}{r_k}, \qquad z_k^4 = \cos\left(\frac{r_k}{2N_k}\right). \tag{4.22}$$

In terms of these parametrizations we find

$$S^+_{ij}(c) = \frac{\sin^2(r_i/2N_i)\sin^2(r_j/2N_j)}{r_i^2 r_j^2} F_{ij}^2, \tag{4.23}$$

$$S^+_{i0}(c) = S^+_{0i}(c) = \sum_{\lambda=1}^{4} \left(z_i^\lambda(t+1) - z_i^\lambda(t)\right)^2. \tag{4.24}$$

Furthermore, we obtain for S_2 (the piece of the action quadratic in the non-zero momentum fields)

$$S_2=\frac{1}{2}\sum_{\mu,\nu,x}\mathrm{Tr}\Big\{\big[1-\tfrac{1}{2}S^{+}_{\mu\nu}(c(t))\big]\big(D_\nu q_\mu(x)-D_\mu q_\nu(x)\big)^2$$
$$-\tfrac{1}{2}S^{-}_{\mu\nu}(c(t))\big(\big[q_\nu(x+\hat\mu),D_\mu q_\nu(x)\big]-\big[q_\mu(x+\hat\nu),D_\nu q_\mu(x)\big]\big)$$
$$-\tfrac{1}{2}S^{-}_{\mu\nu}(c(t))\big(2\big[q_\mu(x+\hat\nu),q_\nu(x+\hat\mu)\big]+\big[D_\mu q_\nu(x),D_\nu q_\mu(x)\big]\big)\Big\},\tag{4.25}$$

provided the covariant derivative acting on a vector in this equation is modified to

$$D_\mu q_\nu(x)=q_\nu(x+\hat\mu)-U^{(0)}_\mu(t+\hat\nu)^\dagger q_\nu(x)U^{(0)}_\mu(t+\hat\nu).\tag{4.26}$$

(Note that the difference with eq. (4.6) only occurs for $\nu=0$. In the decomposition of eq. (4.4), this modification would not be required [16], but apart from having a slightly more compact notation, the two methods, although differing at intermediate stages, should give the same results for the effective action.)

We can rewrite eq. (4.25) in momentum language if we introduce the momentum eigenstates*

$$q^{(k)}_\mu(x)=\sum_{k_\nu=2\pi n_\nu/N_\nu,\,n_\nu=0}^{N_\nu-1}\exp\left(i\sum_{\lambda=0}^{4}k_\lambda x_\lambda\right)q_\mu(k),\tag{4.27}$$

which are also eigenfunctions of the shift operator

$$q^{(k)}_\mu(x+\hat\nu)=\mathrm{e}^{ik_\nu}q^{(k)}_\mu(x).\tag{4.28}$$

Restricting now to time-independent background fields, one has

$$D_\mu\big(q^{(k)}_\nu\big)(x)=\big\{\mathrm{e}^{ik_\mu}-\exp(-i\,\mathrm{ad}\,c_\mu/N_\mu)\big\}q^{(k)}_\nu(x),$$
$$D^\dagger_\mu\big(q^{(k)}_\nu\big)(x)=\big\{\mathrm{e}^{-ik_\mu}-\exp(i\,\mathrm{ad}\,c_\mu/N_\mu)\big\}q^{(k)}_\nu(x).\tag{4.29}$$

We therefore find

$$S_2+\sum_x\mathrm{Tr}(\chi^2)=\sum_k\mathrm{Tr}\big(q_\mu(-k)\mathscr{W}_{\mu\nu}(c,k)q_\nu(k)\big),\tag{4.30}$$

* From now on k will denote a four-vector, rather than the length of the three-vector $\boldsymbol{k}$.

with

$$\mathscr{W}_{\mu\nu}(c,k)=\delta_{\mu\nu}D_\lambda^\dagger D_\lambda+\left[D_\mu,D_\nu^\dagger\right]+\mathscr{G}_{\mu\nu}^+(c,k)+\mathscr{G}_{\mu\nu}^-(c,k), \tag{4.31}$$

where

$$\mathscr{G}_{\mu\nu}^+(c,k)=-\tfrac{1}{2}\delta_{\mu\nu}\sum_\lambda S_{\mu\lambda}^+D_\lambda^\dagger D_\lambda+\tfrac{1}{2}S_{\mu\nu}^+D_\nu^\dagger D_\mu,$$

$$\mathscr{G}_{\mu\nu}^-(c,k)=\tfrac{1}{4}\delta_{\mu\nu}\sum_\lambda\left\{\exp(ik_\lambda+i\,\mathrm{ad}\,c_\lambda/N_\lambda)\,\mathrm{ad}\,S_{\lambda\nu}^-+\mathrm{ad}\,S_{\mu\lambda}^-\exp(-ik_\lambda-i\,\mathrm{ad}\,c_\lambda/N_\lambda)\right\}$$

$$+\tfrac{1}{2}e^{ik_\mu}\,\mathrm{ad}\,S_{\mu\nu}^-\,e^{-ik_\nu}-\tfrac{1}{4}D_\nu^\dagger\,\mathrm{ad}\,S_{\mu\nu}^-D_\mu. \tag{4.32}$$

Like in the continuum [6] we can now compute the effective potential for a time-independent background field. In lattice units one has*

$$V_1(c)=\lim_{N_0\to\infty}-\frac{1}{N_0}\frac{\det\mathscr{M}(c)}{\sqrt{\det\mathscr{W}_{\mu\nu}(c)}}$$

$$=-\int_0^\infty\frac{ds}{s}e^{-2s}I_0(2s)\sum_{k\neq 0}\left\{\tfrac{1}{2}\,\mathrm{tr}\,\mathrm{Tr}\left[\exp\left(-s\mathscr{W}_{\mu\nu}^{(\mathrm{sp})}(c,k)\right)\right]\right.$$

$$\left.-\mathrm{Tr}\left[\exp\left(-s\mathscr{M}^{(\mathrm{sp})}(c,k)\right)\right]\right\}. \tag{4.33}$$

Here we have used

$$\mathscr{W}_{\mu\nu}(c,k)=4\sin^2(k_0/2)\delta_{\mu\nu}+\mathscr{W}_{\mu\nu}^{(\mathrm{sp})}(c,k),\qquad\mathscr{M}(c,k)=4\sin^2(k_0/2)+\mathscr{M}^{(\mathrm{sp})}(c,k). \tag{4.34}$$

where the spatial parts $\mathscr{W}_{\mu\nu}^{(\mathrm{sp})}(c,k)$ and $\mathscr{M}^{(\mathrm{sp})}(c,k)$ only depend on the spatial momenta. Furthermore, this allows one to explicitly perform the sum over the momenta k_0,

$$\lim_{N_0\to\infty}\frac{1}{N_0}\sum_{n_0=1}^{N_0}\exp\left(-4s\sin^2(\pi n_0/N_0)\right)=e^{-2s}I_0(2s), \tag{4.35}$$

where $I_n(z)$ are the modified spherical Bessel functions, which are the Fourier coefficients for the function $e^{z\cos\theta}$. This implies the more general result (which will

* The trace w.r.t. the space-time indices is denoted by tr.

be of use further on)

$$f_s(t,N)=\frac{1}{N}\sum_{n}^{N}\exp(\ 4s\sin^2(\pi(n+t)/N))=\mathrm{e}^{-2s}\sum_{k\in\mathbb{Z}}I_{N|k|}(s)\mathrm{e}^{-2\pi ikt},\tag{4.36}$$

which is easily derived using Poisson resummation.

It is instructive to first evaluate $V_1(c)$ for abelian background fields, for which S^+ and S^- vanish and $\mathcal{W}_{\mu\nu}^{(\mathrm{sp})}=\delta_{\mu\nu}\mathcal{M}^{(\mathrm{sp})}$. Furthermore, $\mathcal{M}^{(\mathrm{sp})}$ has for $c_i^a=r_i\delta_{a3}$ the eigenvalues

$$\omega_\alpha^2(\boldsymbol{r},\boldsymbol{k})=\sum_{j=1}^{3}4\sin^2\left(\frac{k_j+\alpha r_jN_j^{-1}}{2}\right),\qquad \alpha=0,\pm1.\tag{4.37}$$

Therefore

$$V_1^{ab}(\boldsymbol{r})=-\int_0^\infty\frac{\mathrm{d}s}{s}\mathrm{e}^{-2s}I_0(2s)\sum_{\alpha,\boldsymbol{n}}\exp\left[-4s\sum_{j=1}^{3}\sin^2\left(\frac{2\pi n_j+\alpha r_j}{2N_j}\right)\right].\tag{4.38}$$

The s integration can be computed exactly. Again, for later use, we consider the more general integral

$$F=-\int_0^\infty\frac{\mathrm{d}s}{s}\mathrm{e}^{-s\omega^2}f_s(0,N)=-\sum_{k\in\mathbb{Z}}\int_0^\infty\frac{\mathrm{d}s}{s}\mathrm{e}^{-(2+\omega^2)s}I_{N|k|}(2s)\tag{4.39}$$

and use that the Laplace transform of $I_n(z)$ is also a simple function

$$\int_0^\infty\mathrm{d}s\,\mathrm{e}^{-s\cosh(\Omega)}I_n(s)=\frac{\mathrm{e}^{-n\Omega}}{2\sinh(\Omega)}.\tag{4.40}$$

This shows that F is exactly the free energy of a harmonic oscillator with frequency Ω in a heat bath of inverse temperature N. The frequency Ω is related to ω by

$$\omega=2\sinh(\Omega/2).\tag{4.41}$$

More explicitly one has

$$\begin{aligned}\frac{\partial F}{\partial\Omega}&=\cosh(\Omega/2)\frac{\partial F}{\partial\omega}=\sinh(\Omega)\sum_{k\in\mathbb{Z}}\int_0^\infty\mathrm{d}s\,\mathrm{e}^{-(1+\omega^2/2)s}I_{N|k|}(s)\\&=\tfrac{1}{2}\sum_{k\in\mathbb{Z}}\mathrm{e}^{-N|k|\Omega}=-\frac{1}{N}\frac{\partial}{\partial\Omega}\ln\left\{\sum_{n=0}^{\infty}\mathrm{e}^{-N(n+1/2)\Omega}\right\}.\end{aligned}\tag{4.42}$$

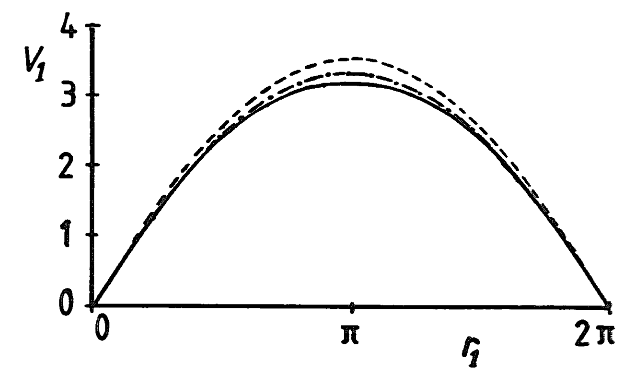


Fig. 2. The effective potential $N(V_1^{ab}(\boldsymbol{r};N)+4\,\mathrm{asinh}\{\sqrt{\sum_i \sin^2(r_i/2N)}\})$, see eq. (4.43), along the r_1 axis for $N=4$ (dashed line), $N=6$ (dashed-dotted line) and $N=\infty$ (solid line), where the latter is equivalent to the continuum.

Therefore, in lattice units (up to an irrelevant additive constant) we find

$$V_1^{ab}(\boldsymbol{r}) = 4\sum_{n_i=0,\,\boldsymbol{n}\neq\boldsymbol{0}}^{N_i-1} \mathrm{asinh}\left\{\sqrt{\sum_{j=1}^{3}\sin^2\left(\frac{2\pi n_j - r_j}{2N_j}\right)}\right\}, \qquad (4.43)$$

whose Taylor expansion around $\boldsymbol{r}=\boldsymbol{0}$ will give the coefficients γ_i as a function of N_1, N_2 and N_3. In fig. 2 we have plotted this one-loop effective potential (including the zero-mode contribution $4\,\mathrm{asinh}\{\sqrt{\sum_i \sin^2(r_i/2N_i)}\}$) for abelian background fields and for cubic lattices with $N=4$, 6 and ∞, where $N=\infty$ corresponds to the continuum result [6].

To determine the coefficients α_i $(i\neq 1)$ we write

$$\mathscr{W}_{\mu\nu}^{(\mathrm{sp})}(c,\boldsymbol{k}) = \delta_0(\boldsymbol{k})\delta_{\mu\nu} + \Delta\mathscr{W}_{\mu\nu}(c,\boldsymbol{k}), \qquad \mathscr{M}^{(\mathrm{sp})}(c,\boldsymbol{k}) = \delta_0(\boldsymbol{k}) + \Delta\mathscr{M}(c,\boldsymbol{k}),$$

$$\delta_0(\boldsymbol{k}) = \sum_{i=1}^{3} 4\sin^2(k_i/2), \qquad (4.44)$$

where $\Delta\mathscr{W}_{\mu\nu}(c,\boldsymbol{k})$ and $\Delta\mathscr{M}(c,\boldsymbol{k})$ vanish for $c=0$. We can now expand the exponentials in eq. (4.33) in powers of c and express the s integrations in terms of the coefficients

$$\delta_n(\boldsymbol{k}) = \int_0^\infty \mathrm{d}s\, s^{n-1}\,\mathrm{e}^{-(2+\delta_0(\boldsymbol{k}))s} I_0(2s) = \tfrac{1}{2}(-\mathrm{d}/\mathrm{d}x)^{n-1}(x+x^2/4)^{-1/2}\Big|_{x=\delta_0(\boldsymbol{k})}, \qquad n>0. \quad (4.45)$$

We have used the algebraic programme FORM [13], independently from calculations by hand, to obtain the results of table 3, which expresses the coefficients α_i

and γ_i as finite lattice momentum sums. For illustration we have only presented the results for a cubic spatial lattice, where $N_1 = N_2 = N_3 = N$. For asymmetric lattices the analogous coefficients can be obtained on request from the author in the form of a subroutine (written in C), which evaluates these coefficients.

Finally we discuss how one computes the coefficient α_1. Like in the continuum [6], this is most easily done by a standard one-loop background field calculation [19] for the term quadratic in c. In this case the external c lines will carry a momentum p_0. Apart from the diagrams that contribute in the continuum, there are now additional diagrams due to terms that would vanish in the naive continuum limit. The best way to go about is to expand S_2 up to the second order in c (for a time-dependent background field). Using eq. (4.25) with*

$$S^-_{0j} = 2i\partial_0 c_j(t)/N_j + \mathrm{O}(c^2), \qquad S^-_{ij} = \mathrm{O}(c^2),$$

$$S^+_{0j} = \tfrac{1}{4}\left(\partial_0 c^a_j(t)\right)^2 N_j^{-2}, \qquad S^+_{ij} = \mathrm{O}(c^4),$$

and adding $\sum_x \mathrm{Tr}(\chi^2)$ to the action, a careful analysis will give (the summation over x, μ, ν and j is implicit)

$$\begin{aligned}
S_2 + \sum_x \mathrm{Tr}(\chi^2) = {}& \mathrm{Tr}\left\{\left(\partial_\mu q_\nu(x)\right)^2\right\} + \mathrm{Tr}\left\{2i\partial_j q_\nu(x)\left[\frac{c_j(t+\hat\nu)}{N_j}, q_\nu(x)\right]\right\}_{(1)} \\
& + \mathrm{Tr}\left\{(\partial_j + 1)q_\nu(x)\,\mathrm{ad}^2\frac{c_j(t+\hat\nu)}{N_j} q_\nu(x)\right\}_{(2)} \\
& + \mathrm{Tr}\left\{iq_0(x)\left[\frac{\partial_0 c_j(t)}{N_j}, \partial_j q_0(x)\right] + i\partial_0 q_j(x)\left[\frac{\partial_0 c_j(t)}{N_j} q_j(x)\right]\right. \\
& \left. + i(2+\partial_0)q_j(x)\left[\frac{\partial_0 c_j(t)}{N_j}, (2+\partial_j)q_0(x)\right]\right\}_{(3)} \\
& + \mathrm{Tr}\left\{\frac{\partial_0 c(t)}{N_j}\left[q_0(x+\hat j) - \partial_0 q_j(x), \left[\frac{c_j(t+1)}{N_j}, q_0(x)\right]\right]\right. \\
& \left. - \frac{1}{8}\left(\frac{\partial_0 c^a_j(t)}{N_j}\right)^2 \left(\partial_0 q_j(x) - \partial_j q_0(x)\right)^2\right\}_{(4)},
\end{aligned} \tag{4.46}$$

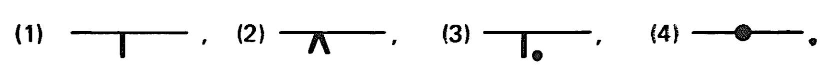
The indices in the brackets denote the various vertices occurring in fig. 3 (this

* The lattice derivative is denoted by ∂_μ, with $\partial_\mu f(x) = f(x+\hat\mu) - f(x)$.

TABLE 3
The coefficients for the lattice in the non-local gauge with a time-independent background field. The coefficients are to be summed over all non-zero lattice momenta ($\sum_{k_i=0}^{N-1}$, excluding $\boldsymbol{k}=\boldsymbol{0}$). We used the shorthand notation $d_i=\delta_i(\boldsymbol{k})$, $s_i=\sin(k_i)$ and $c_i=\cos(k_i)$

	Non-local gauge lattice coefficients
$\alpha_1(N)$	$\{(16d_0(1+d_0/4)d_4/3+4d_2-6(2+d_0)d_3)s_1^2-d_2(3s_1^2+4(1+c_1))$
	$-(1+2d_1d_0)/8+c_1(2d_1-d_2d_0)+d_2d_0^2/4\}N^{-3}$
$\alpha_2(N)$	$-4\{d_1(c_1+d_0/8)/2+d_2(1+c_1)(1+c_2)/2-d_2s_1^2/2-8d_4(s_1s_2)^2/3$
	$-d_2c_1c_2/2+2d_3s_1^2c_2\}N^{-3}$
$\alpha_3(N)$	$\{c_1(c_2+4)d_2/16-(1+c_1)s_2^2d_3/2+(1+c_1+c_2)c_3d_3/4$
	$-(1+c_1+c_2-4c_1c_2-2s_1^2)s_3^2d_4/3-4(s_1s_2)^2c_3d_5+16(s_1s_2s_3)^2d_6/5\}N^{-5}$
$\alpha_4(N)$	$\{(22c_1+8c_1^2-3+6c_1c_2)d_2/24-(18-c_2+9c_1)s_1^2d_3/12+(3+c_1)d_1/48$
	$-2s_1^2c_2(s_1^2+9s_2^2)d_5/3+(6-25s_1^2+3s_2^2-6c_1-6c_2+12c_1c_2)s_2^2d_4/9$
	$+(1+c_1+c_2)c_1d_3/2+32s_1^4s_2^2d_6/15\}N^{-5}-2\alpha_3(N)$
$\alpha_5(N)$	$\{3d_2c_2c_1/8-(c_1c_2c_3-(1+c_1)(1+c_2)(1+4c_3))d_3/2-4(1+c_1)(1+c_2)s_3^2d_4$
	$+4(s_1s_2)^2c_3d_5-64(s_1s_2s_3)^2d_6/15-2(s_1s_2)^2d_4\}N^{-5}$
$\gamma_1(N)$	$-\{4d_2s_1^2-2d_1c_1\}N^{-1}$
$\gamma_2(N)$	$-\{d_1c_1/6-4d_2s_1^2/3+d_2c_1^2-4d_3s_1^2c_1+4d_4s_1^4/3\}N^{-3}$
$\gamma_3(N)$	$-2\{d_2c_1c_2-2d_3(s_1^2c_2+s_2^2c_1)+4d_4(s_1s_2)^2\}N^{-3}$
$\gamma_4(N)$	$\{c_1^2d_2/6-5c_1s_1^2d_3/3+8s_1^4d_4/9+c_1^3d_3/3+4s_1^4c_1d_5/3$
	$-8s_1^6d_6/45-2(c_1s_1)^2d_4-8s_1^2d_2/45+c_1d_1/180\}N^{-5}$
$\gamma_5(N)$	$\{c_1c_2d_2/6+c_2(c_1^2-5s_1^2/3)d_3+2s_2^2d_4(4s_1^2/3-c_1(c_1+2c_2))$
	$+4s_1^2c_2d_5(2s_2^2+s_1^2/3)-8s_1^4s_2^2d_6/3\}N^{-5}$
$\gamma_6(N)$	$\{8(s_1s_2)^2(3c_3d_5-2s_3^2d_6)+2c_1c_2c_3d_3-12s_1^2c_2c_3d_4\}N^{-5}$

figure also includes the obvious ghost diagrams)



We leave it as an exercise to find the appropriate expressions from eq. (4.46) for these vertices. We now wish to extract the term proportional to $\frac{1}{2}(\partial_0 c_i^a)^2$, which will give us $\alpha_1^{(i)}$ (in a cubic volume this is independent of i). Note that the last diagram of fig. 3 is already of this form, by using

$$\sum_t \mathrm{Tr}\Big(c_j(t+1)\big(c_j(t+1)-c_j(t)\big)=\tfrac{1}{2}\sum_t \mathrm{Tr}\big(c_j(t+1)-c_j(t)\big)^2\Big),$$

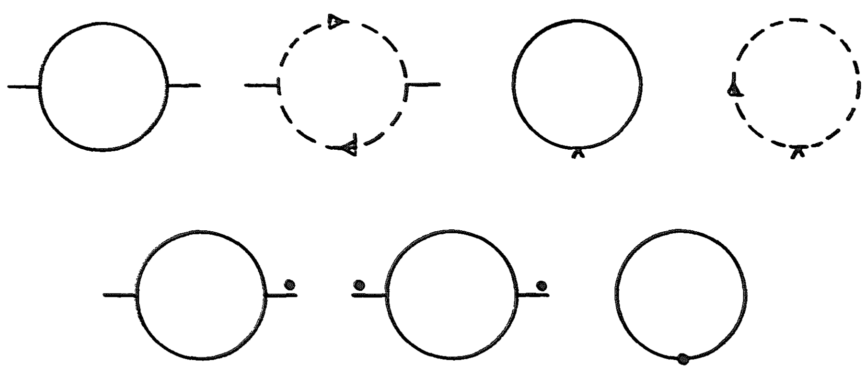


Fig. 3. The one-loop Feynman diagrams that contribute to $\alpha_1^{(i)}(N)$, eq. (4.46).

and has no external momentum flowing through the diagram. Also the third and fourth diagram in fig. 3 are of the tadpole type, but their result is proportional to $\Sigma_t\, c_j^a(t)^2$ and hence they do not contribute to α_1. The remaining diagrams depend more non-trivially on the external momentum and we extract the term proportional to $(\partial_0 c_i^a)^2$ by an expansion in the external momentum p_0. This is most easily done by writing $c_i^a(t) = \sqrt{(2/N_0)}\,\sin(p_0 t) c_i^a$, expanding the result in powers of $\varepsilon \equiv 2\sin(p_0/2)$, since $\Sigma_t(\partial_0 c_i^a(t))^2 = \varepsilon^2 (c_i^a)^2$. As an illustration, we will only exhibit the contribution to $\alpha_1^{(i)}$ from the first two diagrams, (which is the only case where one needs to expand a propagator)

$$\delta\alpha_1^{(i)} = \frac{-4}{\prod_{\mu=0}^{3} N_\mu}\frac{\partial^2}{\partial\varepsilon^2}\sum_k \frac{\sin^2(k_i)}{\left(4\sin^2((k_0+p_0)/2)+\delta_0(\boldsymbol{k})\right)\left(4\sin^2(k_0/2)+\delta_0(\boldsymbol{k})\right)}$$

$$= \frac{1}{\prod_{j=1}^{3} N_j}\sum_{\boldsymbol{k}\neq\boldsymbol{0}} \sin^2(k_i)\big\{4\delta_2(\boldsymbol{k}) - 6(2+\delta_0(\boldsymbol{k}))\delta_3(\boldsymbol{k})$$

$$+ \tfrac{4}{3}\delta_0(\boldsymbol{k})(4+\delta_0(\boldsymbol{k}))\,\delta_4(\boldsymbol{k})\big\}. \qquad (4.47)$$

The final result for α_1 in the case of a cubic lattice can be found in table 3.

The scaling limit corresponds to taking N to infinity and simultaneously g_0 to zero, such that $g_0^{-2} + \alpha_{1,2}$ remains finite. To be more precise one can show that $\alpha_{1,2}(N)$ diverges as $-(11/12\pi^2)\ln(N)$ for large N, in accordance with the (universal) one-loop beta function. If we define

$$\alpha_{1,2}^{\mathrm{R}}(N) = \alpha_{1,2}(N) + \frac{11}{12\pi^2}\ln(N\Lambda_{\mathrm{L}}/\Lambda_{\mathrm{MS}}), \qquad (4.48)$$

with [16, 20] $\ln(\Lambda_{\mathrm{L}}/\Lambda_{\mathrm{MS}}) = -(12\pi^2/11)\times 0.1866792$, $\alpha_{1,2}^{\mathrm{R}}(N)$ has a finite scaling limit which coincides to a high accuracy with the renormalized values in the continuum (see eq. (3.44))

$$\lim_{N\to\infty}\alpha^{R}_{1,2}(N)=\lim_{d\to 3}\alpha^{R}_{1,2}(d)\equiv\lim_{d\to 3}\left(\alpha_{1,2}(d)-\frac{11}{12\pi^2(d-3)}\right). \tag{4.49}$$

Equivalently, comparing the lattice and continuum results give two independent checks on the relation between the lattice and continuum (for dimensional regularization with minimal subtraction) scale parameters [16, 20]. We refer to fig. 1 of the first reference in [4] and sect. 8 for the behaviour of the coefficients as a function of N, especially demonstrating that for $N\to\infty$ the coefficients approach their continuum values as N^{-2}.

5. From transfer matrix to hamiltonian

We have computed in the previous section the effective action (using a time-independent background field to compute the effective potential to one-loop order) and found

$$L_{\rm eff}(c(t))=2N_1N_2N_3\sum_{i=1}^{3}\left(g_0^{-2}+\alpha_1^{(i)}(N)\right)\times\sum_{\mu=1}^{4}\left(z_i^{\mu}(t+1)-z_i^{\mu}(t)\right)^2+\tfrac{1}{2}\left[V_1(c(t+1))+V_1(c(t))\right]. \tag{5.1}$$

where z_i^{μ} are the stereographic coordinates of eq. (4.22). However, time is still discrete. The aim of this section is to show how one computes the effective hamiltonian $\mathscr{H}$ in terms of the transfer matrix $\mathscr{T}$ defined by ($\mathscr{N}$ is a normalization constant)

$$\psi(c;t+1)=\mathscr{T}\psi(c;t)=\mathscr{N}\int \mathrm{d}c'\exp\left[-L_{\rm eff}(c(t+1)=c,c(t)=c')\right]\psi(c';t). \tag{5.2}$$

All we basically need to do is to find an operator representation of $\mathscr{T}$ and take its logarithm. The spectrum of the hamiltonian $\mathscr{H}$ thus obtained is precisely what will be measured in a Monte Carlo analysis using time–time correlation functions on a lattice that (for all practical purposes) extends to infinity in the time direction [3].

It is instructive, as usual, to first consider a simple harmonic oscillator

$$L(t)=\tfrac{1}{2}m\left(x(t+1)-x(t)\right)^2+\tfrac{1}{4}m\omega^2\left(x^2(t+1)+x^2(t)\right). \tag{5.3}$$

Introducing the position and momentum eigenstates $|x\rangle$, resp. $|p\rangle$, such that $\langle x|p\rangle = e^{ipx}$, one finds

$$\mathcal{T}|\psi\rangle = \left\{\int dx' \int dx\, \mathcal{N}|x'\rangle\langle x|\exp\left[-\tfrac{1}{2}m(x'-x)^2 + \tfrac{1}{4}m(x^2+x'^2)\right]\right\}|\psi\rangle. \quad (5.4)$$

Inserting

$$\mathcal{N}\exp\left[-\tfrac{1}{2}m(x'-x)^2\right] = \int dp \exp\left(-\frac{p^2}{2m}\right)\langle x'|p\rangle\langle p|x\rangle, \quad (5.5)$$

with

$$\mathcal{N} = \sqrt{2\pi m} \quad (5.6)$$

we find ($\hat{x}$ and $\hat{p}$ are the position and momentum operators)

$$\mathcal{T} = \exp\left(-\tfrac{1}{4}m\omega^2\hat{x}^2\right)\exp\left(-\frac{\hat{p}^2}{2m}\right)\exp\left(-\tfrac{1}{4}m\omega^2\hat{x}^2\right) = \exp(-\mathcal{H}) \quad (5.7)$$

and one evaluates $\mathcal{H}$ using the Cambell–Baker–Hausdorff (CBH) formula. It should not come as a surprise that we can compute $\mathcal{H}$ exactly for the harmonic oscillator.

$$\mathcal{H} = \frac{\hat{p}^2}{2M} + \tfrac{1}{2}M\Omega^2\hat{x}^2, \qquad M = \frac{\sinh(\Omega)m}{\Omega}, \qquad \omega = 2\sinh(\Omega/2). \quad (5.8)$$

That $\mathcal{H}$ is again harmonic can be seen either by using the CBH formula or the fact that, as we have shown in sect. 4 (see eq. (4.42)), the partition function $Z = e^{-NF} = \mathrm{Tr}(\mathcal{T}^N)$ is that of a harmonic oscillator with frequency Ω. One can then compute M from the identity

$$M^{-1} = 2\Omega\langle 0|\hat{x}^2|0\rangle = \lim_{N\to\infty} -\frac{1}{m\omega}\frac{\partial \ln(Z)}{\partial\omega}.$$

As long as we are only interested in the spectrum of the transfer matrix and not in the eigenfunctions, we could equally well have defined

$$\mathcal{T}' = \exp(-\mathcal{H}') = \exp\left(-\frac{\hat{p}^2}{4m}\right)\exp\left(-\tfrac{1}{2}m\omega^2\hat{c}^2\right)\exp\left(-\frac{\hat{p}^2}{4m}\right), \quad (5.9)$$

since $\mathrm{Tr}(\mathcal{T}^{N_0}) = \mathrm{Tr}(\mathcal{T}'^{N_0})$. For the harmonic oscillator $\mathcal{H}'$ is related to $\mathcal{H}$ by the canonical transformation $\hat{p} \to -\omega m\hat{x}, \hat{x} \to \hat{p}/\omega m$ such that

$$\mathscr{H}' = \frac{\hat{p}^2}{2M'} + \tfrac{1}{2}M'\Omega^2\hat{x}^2, \qquad M' = \frac{2\tanh(\Omega/2)m}{\Omega}, \tag{5.10}$$

and $\mathscr{H}'$ is equivalent to $\mathscr{H}$ by a unitary transformation. Note that in the continuum limit

$$\lim_{a\to 0} a^{-1}\mathscr{H}(a^{-1}m, a\omega) = \lim_{a\to 0} a^{-1}\mathscr{H}'(a^{-1}m, a\omega) = \frac{\hat{p}^2}{2m} + \tfrac{1}{2}m\omega^2\hat{x}^2. \tag{5.11}$$

It is quite straightforward to generalize the construction of the hamiltonian to

$$L_{\text{eff}}(t) = \tfrac{1}{2}\sum_i m_i(x_i(t+1) - x_i(t))^2 + \tfrac{1}{2}[V(\boldsymbol{x}(t+1)) + V(\boldsymbol{x}(t))]. \tag{5.12}$$

We find

$$\mathscr{H} = -\ln\left\{\exp(-\tfrac{1}{2}V(\boldsymbol{x}))\exp\left(-\sum_i \frac{p_i^2}{2m_i}\right)\exp(-\tfrac{1}{2}V(\boldsymbol{x}))\right\}. \tag{5.13}$$

Apart from the constraint $\Sigma_{\mu=1}^4 z_i^\mu z_i^\mu = 1$, our effective lagrangian (5.1) is of the above form. One easily convinces oneself that this constraint, supplemented with the constraint that the radial component of the momentum vanishes, does not interfere with the construction of the transfer matrix in an operator formulation. All we have to do is to replace p^2 in eq. (5.13) with the laplacian Δ_3 on S^3, such that

$$\mathscr{H}_{\text{latt}} = -\ln\left\{\exp(-\tfrac{1}{2}V_1(c))\exp\left(-\tfrac{1}{2}\sum_i \frac{\Delta_3^{(i)}}{m_i}\right)\exp(-\tfrac{1}{2}V_1(c))\right\},$$

$$\Delta_3^{(j)} = -\frac{4N_j^2}{\sin^2(r_j/2N_j)}\frac{\partial}{\partial r_j}\sin^2\left(\frac{r_j}{2N_j}\right)\frac{\partial}{\partial r_j} + \frac{L_{(j)}^2}{\sin^2(r_j/2N_j)},$$

$$m_j = 4N_1N_2N_3(g_0^{-2} + \alpha_1^{(j)}(N)). \tag{5.14}$$

Wavefunctions are normalized w.r.t. the L^2 norm on $(S^3)^3$, which we conveniently transform by a stereographic projection to the L^2 norm on $\mathbb{R}^9$. Thus, rescaling the wavefunctions with a factor

$$J(c)=\prod_{i=1}^{3}J_i(c), \qquad J_i(c)=\frac{\sin(r_i/(2N_i))}{r_i/(2N_i)}, \tag{5.15}$$

transforms $\Delta_3^{(i)}$ to

$$\tilde{\Delta}_3^{(i)}=J(c)\Delta_3^{(i)}J(c)^{-1}=4N_i^2\left\{-\frac{\partial^2}{\partial c_i^{a2}}+\left(J_i(c)^{-2}-1\right)\frac{L_{(i)}^2}{r_i^2}-\frac{3}{4N_i^2}\right\}. \tag{5.16}$$

The continuum limit is obtained by writing $L_i=aN_i$, $\mathscr{H}=a^{-1}\mathscr{H}_{\text{latt}}$ and taking the scaling limit $a\to 0$, $g_0\to 0$, $N_i\to\infty$, such that L_i and m_i remain finite. This implies that the CBH formula for eq. (5.14) will give an expansion in $a\sim N^{-1}$, of the form

$$\mathscr{H}=-a^{-1}\ln\{\mathrm{e}^{-aA}\,\mathrm{e}^{-2aB}\,\mathrm{e}^{-aA}\}. \tag{5.17}$$

Since we will consider N as small as 4, we will expand $\mathscr{H}$ to fourth order in a. Furthermore, we use the freedom to choose unitary transformations to bring $\mathscr{H}$ to its simplest possible form. Define

$$F(A,B)=-a^{-1}\ln\{\mathrm{e}^{-aA}\,\mathrm{e}^{-aB}\}, \tag{5.18}$$

then

$$\mathscr{H}=-a^{-1}\ln\{\mathrm{e}^{-aF(A,B)}\,\mathrm{e}^{-aF(B,A)}\}=H_1(A,B)+H_2(A,B),$$
$$H_1(A,B)=H_1(B,A), \qquad H_2(A,B)=-H_2(B,A). \tag{5.19}$$

Explicitly to $\mathrm{O}(a^4)$

$$\begin{aligned}H_1(A,B)&=2(A+B)+\frac{a^2}{6}[A-B,[A,B]]\\&\quad-\frac{a^4}{360}([A,[B,[B,[A,B]]]]-[B,[A,[A,[A,B]]]])\\&\quad-\frac{a^4}{360}(\mathrm{ad}^3(A-B)+8\,\mathrm{ad}(A-B)\mathrm{ad}\,A\,\mathrm{ad}\,B\\&\qquad+30\,\mathrm{ad}[A,B]\mathrm{ad}(A+B))[A,B],\\H_2(A,B)&=-\frac{a^2}{2}[A+B,[A,B]]+\frac{a^4}{24}(\mathrm{ad}^3(A+B)+\mathrm{ad}(A+B)\mathrm{ad}\,A\,\mathrm{ad}\,B\\&\qquad+\mathrm{ad}[A,B]\mathrm{ad}(A-B))[A,B].\end{aligned} \tag{5.20}$$

On general grounds we know that there exists a unitary transformation $U=\exp(X)$ such that

$$H_1(A,B)+H_2(A,B)=e^{X(A,B)}(H_1(A,B)-H_2(A,B))e^{-X(A,B)},$$
$$X(A,B)=-X^\dagger(A,B)=-X(B,A). \tag{5.21}$$

This X is therefore determined implicitly by the equation

$$H_2(A,B)=\tanh\left(\frac{\operatorname{ad}X(A,B)}{2}\right)H_1(A,B). \tag{5.22}$$

To lowest order one easily finds $X(A,B)=(a^2/2)[A,B]$. We use it to define the symmetric function

$$H_s(A,B)=e^{-X(A,B)/2}(H_1(A,B)+H_2(A,B))e^{X(A,B)/2}$$
$$=\left(1-\tfrac{1}{8}\operatorname{ad}^2X(A,B)\right)H_1(A,B)+O(a^6). \tag{5.23}$$

Finally we wish to minimize the presence of terms that mix coordinates and momenta by applying a unitary transformation $\exp(Y)$

$$Y(A,B)=-\frac{a^2}{12}[A,B]+\frac{a^4}{720}[B,[B,[A,B]]]+\frac{a^4}{240}[B,[A,[A,B]]], \tag{5.24}$$

which transforms the effective hamiltonian to

$$\mathcal{H}=2(A+B)+\frac{a^2}{3}[A,[A,B]]-\frac{a^4}{15}[A,[A,[B,[A,B]]]]$$
$$-\frac{a^4}{45}[[A,B],[B,[A,B]]]-\frac{a^4}{360}[A,[A,[A,[A,B]]]]$$
$$+\frac{17a^4}{360}[B,[A,[A,[A,B]]]]+O(a^6),$$
$$A=\frac{1}{2a}V_1(c),\qquad B=\sum_i\frac{\tilde{\Delta}_3^{(i)}}{4m_ia}. \tag{5.25}$$

The last two terms will vanish and the fourth term that mixes coordinates and momenta cannot be entirely transformed away. As we have not included similar terms in the effective lagrangian, we will also here ignore them. Thus in lattice units one finds

$$\hat{\mathscr{H}}_{\text{latt}} = \Delta(c) + V_1(c) - \tfrac{1}{24}[V_1(c),[\Delta(c),V_1(c)]] + \tfrac{1}{480}[V_1(c),[V_1(c),[\Delta(c),[\Delta(c),V_1(c)]]]],$$

$$\Delta(c) = \sum_i \frac{\tilde{\Delta}_3^{(i)}}{2m_i}, \tag{5.26}$$

which was expanded to eighth order in c. This modifies the coefficients α_i and γ_i and furthermore introduces extra terms, which for the cubic case are of the form

$$\sum_i r_i^8,\ \sum_{i\neq j} r_i^2 r_j^6,\ \sum_{i>j} r_i^4 r_j^4,\ \prod_i r_i^2 \sum_j r_j^2,\ \sum_{i\neq j} r_i^4 F_{ij}^2,\ \sum_{i\neq j\neq k} r_k^4 F_{ij}^2,$$

$$\sum_{i\neq j} r_i^2 r_j^2 F_{ij}^2,\ \sum_{i\neq j\neq k} r_i^2 r_k^2 F_{ij}^2,\ \sum_{i\neq j}\left(F_{ij}^2\right)^2,\ \sum_{i\neq j\neq k} F_{ij}^2 F_{jk}^2,\ \det{}^2 c \sum_i r_i^2. \tag{5.27}$$

The coefficients including the transformation from the transfer matrix to the effective hamiltonian will for obvious reasons not be displayed here, but can be obtained on request from the author in the form of a C-programme subroutine.

We will not reproduce here the effect of the lattice artifacts on the spectrum and refer the reader to ref. [4] where these issues were adequately discussed. We used the connection between the bare and renormalized coupling constant implied by the relation between Λ_{L} and $\Lambda_{\overline{\mathrm{MS}}}$ [16, 20] to add the two-loop continuum effective potential of eq. (3.49) to V_1 before using eq. (5.26), which should thus include the bulk of the two-loop correction. We expect the overall uncertainty due to ignoring the $\mathrm{O}(g_0^4)$ mixed coordinate and momentum terms and using the continuum rather than lattice two-loop correction to be of the order of 1 to 2 percent.

6. Some two-loop lattice results

In this section we will set up lattice perturbation theory in an abelian time-independent zero-momentum background to two-loop order for a cubic spatial lattice. Here we will prefer the decomposition of eq. (4.4), $U_{x,x+\hat{\mu}} = \hat{U}_\mu(x) U_\mu^{(0)}$. Note that now $U_\mu^{(0)}$ is time independent and $[U_\mu^{(0)}, U_\nu^{(0)}] = 0$. The Wilson action can therefore be written as

$$S = \frac{1}{g_0^2} \sum_{\mu,\nu,x} \mathrm{Tr}\left\{1 - \hat{U}_\mu(x)\left(U_\mu^{(0)}\hat{U}_\nu(x+\hat{\mu})U_\mu^{(0)\dagger}\right)\left(U_\nu^{(0)}\hat{U}_\mu(x+\hat{\nu})U_\nu^{(0)\dagger}\right)\hat{U}_\nu(x)\right\}. \tag{6.1}$$

We will introduce three different parametrizations for $\hat{U}_\mu(x)$

(i) $\hat{U}_\mu(x) = \exp(i\hat{q}_\mu(x)) = 1 + i\hat{q}_\mu(x) - \frac{1}{2}\hat{q}_\mu(x)^2 - \frac{1}{6}i\hat{q}_\mu(x)^3 + \frac{1}{24}\hat{q}_\mu(x)^4 + \dots,$

(ii) $\hat{U}_\mu(x) = \dfrac{(1 + iq_\mu(x)/2)^2}{1 + q_\mu(x)^2/4} = 1 + iq_\mu(x) - \frac{1}{2}q_\mu(x)^2 - \frac{1}{4}iq_\mu(x)^3 + \frac{1}{8}q_\mu(x)^4 + \dots,$

(iii) $\hat{U}_\mu(x) = 1 + ig_0Q_\mu(x) - \frac{1}{2}g_0^2Q_\mu(x)^2 - \frac{1}{8}g_0^4Q_\mu(x)^4 + \dots.$ (6.2)

The first is most convenient from the point of gauge fixing. The second is related to the stereographic coordinates of SU(2) in terms of which the Haar measure is most easily expressed

$$\mathrm{d}^3\hat{U}_\mu(x) = \frac{\mathrm{d}^3q_\mu(x)}{\left(1 + q_\mu(x)^2/4\right)^3} \tag{6.3}$$

and the third coordinate choice is the most convenient for expanding the action to fourth order in q, required for a two-loop analysis. The different parametrizations are related as follows

$$q_\mu(x) = 2\tan\left(\hat{q}_\mu(x)/2\right) = \hat{q}_\mu(x)\left(1 + \tfrac{1}{12}\hat{q}_\mu(x)^2 + \dots\right),$$

$$g_0Q_\mu(x) = q_\mu(x)\left(1 + uq_\mu(x)^2 + \dots\right), \qquad u = -\tfrac{1}{6}. \tag{6.4}$$

We will, however, also consider arbitrary u which is obtained from the results with $u = -\frac{1}{6}$ by the rescaling

$$Q_\mu(x) \to Q_\mu(x)\left(1 + \left(u + \tfrac{1}{6}\right)g_0^2Q_\mu(x)^2 + \dots\right). \tag{6.5}$$

The purpose of having arbitrary u is to have a consistency check. For the same reason we generalize the gauge-fixing function to

$$\hat{\chi}(v) = (1 - P)\hat{D}_\mu^\dagger\left(\hat{q}_\mu(x)\left(1 + v\hat{q}_\mu(x)^2\right)\right). \tag{6.6}$$

Introducing

$$R^{\nu}_{\mu}(x)=U^{(0)}_{\nu}Q_{\mu}(x+\hat{\nu})U^{(0)\dagger}_{\nu}=\left(\hat{D}_{\nu}+1\right)Q_{\mu}(x), \tag{6.7}$$

it is straightforward but tedious to find to $O(g_0^2Q^4)$

$$\begin{aligned}S(u=-\tfrac{1}{6})=\tfrac{1}{2}\sum_{\mu,\nu,x}\mathrm{Tr}\Bigg\{&\Bigg(\hat{D}_{\mu}Q_{\nu}(x)-\hat{D}_{\nu}Q_{\mu}(x)+\frac{ig_0}{2}\left[Q_{\mu}(x),Q_{\nu}(x)\right]\\&+\frac{ig_0}{2}\left[R^{\nu}_{\mu}(x),R^{\mu}_{\nu}(x)\right]\Bigg)^2+\frac{g_0^2}{4}\Big[\left(R^{\nu}_{\mu}(x)+R^{\mu}_{\nu}(x)\right)\left(R^{\nu}_{\mu}(x)\right.\\&\left.-R^{\mu}_{\nu}(x)\right)-\left(Q_{\mu}(x)+Q_{\nu}(x)\right)\left(Q_{\mu}(x)-Q_{\nu}(x)\right)\Big]^2-g_0^2\left(Q_{\mu}(x)\right.\\&\left.-Q_{\nu}(x)\right)\hat{D}_{\mu}Q_{\nu}(x)\left(R^{\nu}_{\mu}(x)-R^{\mu}_{\nu}(x)\right)\hat{D}_{\nu}Q_{\mu}(x)\Bigg\},\end{aligned} \tag{6.8}$$

$$S_{\mathrm{gf}}(u,v)=\sum_{\mu,x}\mathrm{Tr}\{\hat{\chi}^2(v)\}=\sum_{\mu,x}\mathrm{Tr}\left\{\hat{D}^{\dagger}_{\mu}\left(Q_{\mu}(x)\left(1+(v-u)g_0^2Q_{\mu}(x)^2\right)\right)\right\}^2, \tag{6.9}$$

$$S_{\mathrm{m}}(u,w)=g_0^2\left(\tfrac{5}{2}u+\tfrac{1}{6}+w\right)\sum_{\mu,x}\mathrm{Tr}\left\{Q_{\mu}(x)^2\right\}, \tag{6.10}$$

where $S_{\mathrm{m}}(u,w)$ is the contribution due to the Haar measure [20] and the jacobian associated to changing d^3q into d^3Q. The value of w will in general be zero, but for the special case that we have imposed the constraint $P\hat{q}_{\mu}=0$, one will pick up an additional jacobian factor

$$\int d^3Q_{\mu}(x)\delta(P\hat{q})=\delta(PQ_{\mu})\left(1+\frac{5ug_0^2}{2N^3}\sum_{\mu,x}\mathrm{Tr}\left(Q_{\mu}(x)^2\right)+\dots\right), \tag{6.11}$$

such that in the presence of the constraint $P\hat{q}=0$

$$w=-5u/2N^3,\qquad (P\hat{q}=0). \tag{6.12}$$

There will be two more contributions, namely

$$\Delta S(u) \equiv S(u) - S(u = -\tfrac{1}{6}) = 2g_0^2(u + \tfrac{1}{6}) \sum_{\mu,\nu,x} \mathrm{Tr}\{(\hat{D}_\mu Q_\nu(x) - \hat{D}_\nu Q_\mu(x))\hat{D}_\mu Q_\nu(x)^3\}, \tag{6.13}$$

$$S_{\mathrm{gh}}(v) = \sum_{\mu,x} \mathrm{Tr}\Big\{\bar{\psi}(x)\hat{\mathscr{M}}\psi(x) + g_0^2 v\bar{\psi}(x)\hat{D}_\mu^\dagger\Big(Q_\mu(x)^2\hat{D}_\mu\psi(x) + Q_\mu(x)\mathrm{Tr}\{Q_\mu(x)\hat{D}_\mu\psi(x)\}\Big)\Big\}, \tag{6.14}$$

where $\hat{\mathscr{M}}$ was defined in eq. (4.14) and the term proportional to v follows from

$$\delta_{\cdot 1}\hat{q}_\mu^3(x) = \hat{q}_\mu^2(x)\,\delta_{\cdot 1}\hat{q}_\mu(x) + \hat{q}_\mu(x)\mathrm{Tr}\{\hat{q}_\mu(x)\delta_{\cdot 1}\hat{q}_\mu(x)\}. \tag{6.15}$$

The ghost and vector three-point vertices are still quite manageable, which we will specify for arbitrary abelian background fields

$$U_\mu^{(0)} = \exp(ic_\mu/N_\mu), \qquad c_\mu^a = r_\mu\delta_{a3}(1 - \delta_{\mu 0}). \tag{6.16}$$

As in the continuum, we can introduce $Q_\mu^\alpha(k)$, $\alpha \in \{-1, 0, 1\}$ (see eq. (3.30)), which are eigenstates for the covariant derivatives (k is a lattice momentum)

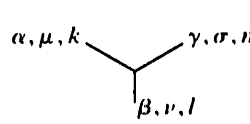
$$\hat{D}_\mu^\dagger Q_\mu^\alpha(k) = (\mathrm{e}^{ik_\mu^{(\alpha)}} - 1)Q_\nu^\alpha(k), \qquad \mathrm{e}^{ik_\mu^{(\alpha)}} \equiv \exp\left[i\left(k_\mu + \frac{\alpha r_\mu}{N_\mu}\right)\right]. \tag{6.17}$$

One easily finds the propagators to be

$$\mu,\alpha \underset{k}{\text{———}} \nu,\beta = \delta_{\mu\nu}\,\delta_{\alpha+\beta,0}\Delta_\alpha(k), \qquad \alpha \underset{k}{\text{- - ->- -}} \beta = \delta_{\alpha+\beta,0}\Delta_\alpha(k),$$

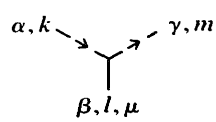
$$\Delta_\alpha^{-1}(k) = \sum_{\mu=0}^{4} 4\sin^2(k_\mu^{(\alpha)}/2), \tag{6.18}$$

whereas the vector three-point vertex is given by ($\varepsilon_{+-0} = 1$)



$$= -i\frac{g_0}{2}\varepsilon_{\alpha\beta\gamma}\Big\{(\mathrm{e}^{im_\sigma^{(\gamma)}} - \mathrm{e}^{ik_\sigma^{(\alpha)}})(1 + \mathrm{e}^{il_\mu^{(\beta)}})\,\delta_{\mu\nu} + \text{cyclic}\Big\} \tag{6.19}$$

and the ghost three-point vertex is found to be



$$= -i\frac{g_0}{2}\varepsilon_{\alpha\beta\gamma}\left(e^{ik_\mu^{(\alpha)}} - 1\right)\left(1 + e^{il_\mu^{(\beta)}}\right). \tag{6.20}$$

(Each n-point function will also carry a factor $(N_0 N^3)^{1-n/2}$.)

The general usefulness of this result is that when the background field and P are put to zero, eqs. (6.18)–(6.20) give the Feynman rules for the Lorentz gauge (see also ref. [20]). For a much more complete and systematic method for setting up lattice perturbation theory the reader is advised to consult ref. [18]. If one wishes one can replace ($\alpha, \beta, \gamma, \varepsilon_{\alpha\beta\gamma}$ and $\delta_{\alpha+\beta,0}$) by ($a, b, c, i\varepsilon_{abc}$ and δ_{ab}) to convert it to a more conventional formulation. In these conventions the Feynman rules are furthermore valid in the presence of a magnetic field generated by twisted boundary conditions [8, 12, 21], provided one defines $k_\mu^{(a)} = k_\mu + \lambda_\mu^{(a)}(m)/N_\mu$, with $\lambda_0^{(a)}(m) = 0$ and $\lambda_i^{(a)}(m)$ as defined in eq. (6c) of ref. [12] (see also ref. [18]). This might be helpful if one wishes to also include lattice artifacts in the spectrum calculations in the presence of twisted boundary conditions [21, 22]. At one loop these lattice artifacts have recently been included [30].

To compute the two-loop effective potential for a time-independent abelian background field (or the two-loop vacuum energy) we have to sum the diagrams of fig. 1 (plus a possible non-local term associated to ghosts when we impose the constraint $P\hat{q} = 0$). The full expression for the four-point vertex is quite complicated, but fortunately we do not really need it since the contribution of the four-point vertex to V_2 is given by

$$N_0 V_2^{(4)} = \left\langle S^{(4)} + S_{\rm gf}^{(4)} + S_{\rm m}^{(2)} + S_{\rm gh}^{(2)} \right\rangle_{\rm c}, \tag{6.21}$$

where $S^{(n)}$ stands for terms proportional to $Q_\mu(x)^n$. Before computing this, we first give the contribution to V_2 due to the three-point vertices (the first two diagrams in fig. 1)

$$V_2^{(3)} = -\frac{g_0^2}{N_0^2 N^3} \sum_{e^{i(k_\mu^{(\alpha)} + l_\mu^{(\beta)} + m_\mu^{(\gamma)})} = 1} \varepsilon_{\alpha\beta\gamma}^2 \Delta_\alpha(k)\Delta_\beta(l)\Delta_\gamma(m)$$
$$\times \left\{ \sum_{\mu\nu} \sin^2\left(\frac{k_\mu^{(\alpha)} - l_\mu^{(\beta)}}{2}\right)\cos^2\left(\frac{m_\nu^{(\gamma)}}{2}\right) + \sum_\mu \left[\sin\left(k_\mu^{(\alpha)}\right)\sin\left(l_\mu^{(\beta)}\right) - \tfrac{1}{2}\sin^2\left(k_\mu^{(\alpha)}\right)\right] \right\}. \tag{6.22}$$

One easily checks that this has the correct naive [23] scaling limit, by comparing with what we found in sect. 2. Unlike in the continuum, it seems impossible to factorize this result in the sum of products of one-loop factors. We will still, however, be able to evaluate the time momentum sums exactly for $N_0 \to \infty$, due to the remarkable formula

$$
\begin{aligned}
&M(m_1^2, m_2^2, m_3^2) \\
&\equiv \int_0^{2\pi} \mathrm{d}k \int_0^{2\pi} \mathrm{d}p \frac{1}{\left(4\sin^2\left(\frac{k_0}{2}\right)+m_1^2\right)\left(4\sin^2\left(\frac{p_0}{2}\right)+m_2^2\right)\left(4\sin^2\left(\frac{k_0+p_0}{2}\right)+m_3^2\right)} \\
&= \frac{\pi^2}{2A}\frac{B+1}{B-1}, \\
&\qquad A = \prod_{i=1}^{3} m_i\sqrt{1+\tfrac{1}{4}m_i^2}, \qquad B = \prod_{i=1}^{3}\left(1+\tfrac{1}{2}m_i^2+m_i\sqrt{1+\tfrac{1}{4}m_i^2}\right). \qquad (6.23)
\end{aligned}
$$

This result is proved most easily in the coordinate representation (I thank G. 't Hooft and J. Groeneveld for suggesting this) because the propagator is as in the continuum exponential in t, which is implicit in eq. (4.42)

$$
\Delta_m(t) = 2\pi\int_0^{2\pi} \mathrm{d}k \frac{e^{ikt}}{4\sin^2(k/2)+m^2} = \frac{\pi\left(1+m^2/2+m\sqrt{1+m^2/4}\right)^{-|t|}}{m\sqrt{1+m^2/4}}. \qquad (6.24)
$$

The result of eq. (6.23) follows now trivially from

$$
M(m_1^2, m_2^2, m_3^2) = \frac{1}{2\pi}\sum_{t\in\mathbb{Z}} \Delta_{m_1}(t)\Delta_{m_2}(t)\Delta_{m_3}(t). \qquad (6.25)
$$

The computation of $\langle S^{(4)} + S^{(4)}_{\mathrm{gf}}\rangle_c$ can be considerably simplified by noting e.g., that terms like $\mathrm{Tr}(R^{\nu}_{\mu}Q^3_{\nu})$ have vanishing expectation values, and that $\Sigma_x \mathrm{Tr}(R^{\nu}_{\mu}(x)^4) = \Sigma_x \mathrm{Tr}(Q_{\mu}(x)^4)$. One finds after tedious algebra

$$
\begin{aligned}
V_2^{(4)} = \frac{g_0^2}{N_0^2N^3}\sum_{k,l}\Bigg\{ & 4\kappa\delta_{k+l,0}\sum_{\alpha\neq\beta}\Delta_\alpha(k)\Delta_\beta(l) + \tfrac{1}{8}\sum_{\mu\neq\nu,\alpha,\beta}\cos(k_\mu^{(\alpha)})\cos(l_\nu^{(\beta)})\Delta_\alpha(k)\Delta_\beta(l) \\
& + \tfrac{3}{4}\sum_{\mu,\alpha,\beta}\left(\cos(k_\mu^{(\alpha)})-\tfrac{1}{2}\right)\Delta_\alpha(k)\Delta_\beta(l) - \tfrac{1}{4}\sum_{\mu\neq\nu,\alpha}\cos(k_\mu^{(\alpha)})\cos(l_\nu^{(\alpha)})\,\Delta_\alpha(k)\Delta_\alpha(l) \\
& + \tfrac{3}{4}\sum_{\mu,\alpha}\sin^2\left(\frac{k_\mu^{(\alpha)}-l_\mu^{(\alpha)}}{2}\right)\Delta_\alpha(k)\Delta_\alpha(l)\Bigg\} - \frac{g_0^2}{N_0}\left(\frac{12\kappa-11\varepsilon}{24N^3}+\tfrac{1}{8}\right)\sum_{\alpha,k}\Delta_\alpha(k). \qquad (6.26)
\end{aligned}
$$

The parameters κ and ε will distinguish the different situations. When we consider the non-local background field gauge $\kappa = 1$ and otherwise it is zero. The parameter $\varepsilon = 1$ corresponds with the fact that the $\boldsymbol{k} = \boldsymbol{0}$ mode is excluded in the momentum sums over $\boldsymbol{k}$. For the case of twisted boundary conditions in the Lorentz gauge one therefore has to take $\varepsilon = \kappa = 0$. It is important to observe that V_2 is independent of u and v. For $\varepsilon = 1$ this confirms the subtlety of eq. (6.11).

Combining $V_2^{(3)}$ and $V_2^{(4)}$ allows us in principle to calculate the lattice artifact contributions to the two-loop effective potential. It should be noted that in the scaling limit $N \to \infty$, NV_2 is supposed to remain finite, but separate terms can diverge as N^4 and subtle cancellations are required. Although we did not discuss this, the same happens for the one-loop coefficients, which are combinations of terms that in general will diverge, but nevertheless when taken together will reproduce the correct result for $N \to \infty$ (see table 3). It is from the lattice calculation that we became aware of the problems that were discussed in sect. 2. Like in sect. 2, let us consider the difference between the non-local and Lorentz gauge results for V_2, computed with a time-independent background field. This is precisely the sum of the terms proportional to κ appearing in eq. (6.26). For a vanishing background field one simply finds

$$V_2^{\mathrm{N}}(0) - V_2^{\mathrm{L}}(0) = N^{-3} \sum_{\boldsymbol{k} \neq \boldsymbol{0}} \left(24\delta_1(\boldsymbol{k})^2 - \tfrac{3}{2}\delta_1(\boldsymbol{k})\right). \tag{6.27}$$

But, this does not vanish as $1/N$, which is required for eq. (5.27) to have a finite scaling limit. It is yet another manifestation of the fact that at two-loop order, one is not allowed to neglect the contributions due to a "dynamical" background field.

Let us conclude this section by mentioning that in the presence of twisted boundary conditions [12], eqs. (6.22) and (6.26) provide the complete result for the two-loop vacuum energy, by putting $\varepsilon = \kappa = 0$ and $\{\alpha, \beta, \gamma\} \in \{1, 2, 3\}$ (see the discussion below eq. (6.20)). Using eq. (6.23) one can express this vacuum energy as a six-dimensional sum over the spatial lattice momenta and NV_2 should now have a finite scaling limit that will coincide with the continuum value [12].

7. Discussion

This paper is intended as the last in a series of papers that studied gauge theories in finite volumes. See ref. [24] for a more pedagogical presentation. The most detailed results are available for SU(2), both from the analytical and the Monte Carlo points of view. This is an important "laboratory" for non-abelian gauge theories, but fortunately analytic results are now also available for the more realistic gauge group SU(3) [25], with results quite similar to those for SU(2). Also massless fermions have been included [26, 27], which for example showed that in small volumes chiral symmetry is unbroken [26], as was recently confirmed in a Monte Carlo analysis [28].

From the theoretical point of view, the significance of the intermediate volume calculations is that one deals with wave functionals that spread out beyond the Gribov horizons closest to the origin in field space, which to some extent dominate the low-lying mass spectrum. For a more thorough discussion see ref. [29], which also discusses the case of arbitrary gauge groups in more detail. It is hoped that our understanding in intermediate volumes, which is by now fairly complete, will give important clues for the non-perturbative dynamics in larger volumes. Still much work remains to be done in that direction.

The relatively simpler case of SU(2) has allowed quite a detailed analysis, including higher-order corrections and lattice artifacts in the analytic calculations. This has been the subject of the present paper. Of practical use will be the new terms of the form $c^2\partial^2/\partial c^2$, that complete the effective hamiltonian (in a finite cubic volume) to fourth order in the coupling constant g (where $c = O(g^{2/3})$). For easy reference, the resulting effective hamiltonian is summarized in sect. 8, together with the numerical values of the various coefficients.

This paper furthermore addresses a few rather technical issues. It gives the details (in sects. 4 and 5, summarized in sect. 8) of the lattice calculations of which the results were presented in ref. [4]. These results demonstrated that after including the lattice effects in the analytic calculations, the agreement with the most accurate Monte Carlo data [2] (with statistical errors as small as 2%, and systematic errors presumably under control) was almost perfect in volumes up to 0.7 fermi. Another motivation to write this paper was to include two-loop lattice corrections* of the type that were also included in the continuum. Sect. 6 demonstrated that such a computation can be performed in principle. However, in the course of this analysis we became aware of an embarrassing flaw in the background field calculations we had employed also in the continuum. It is most apparent when comparing the non-local background field gauge we had used in the past with the Lorentz gauge. In sect. 2 we demonstrated that at two-loop order these two gauges give inequivalent results.

We performed the laborious, but systematic and reliable method of hamiltonian perturbation theory [5] up to fourth order in the gauge coupling constant (sect. 3) in order to understand the origin of this problem. (The help of a powerful algebraic manipulation programme like FORM [13] was essential to bring this calculation to a successful end.) By comparing the hamiltonian and background field methods, the origin of the gauge dependence in the latter arises because that method does not consistently incorporate the dynamical (i.e. time-dependent) nature of the background field. Although we were aware of this, we had assumed

* For a much more general and systematic discussion of lattice perturbation theory the reader should consult refs. [17, 23] and in particular ref. [18].

incorrectly that this only affected the $g_0^4 c^2 e^2$ terms (e is the momentum conjugate to c). To be a little more explicit, the hamiltonian analysis receives contributions from disconnected diagrams, since these diagrams now depend on the operators c and e which do not mutually commute. (If they would commute, disconnected diagrams will not contribute to the effective hamiltonian.) These so-called disconnected contributions are precisely of the form observed in the difference between the Lorentz and non-local background field gauges. They are actually of quite a simple form and it is very unfortunate (and frustrating) not to have been to modify the background field analysis to incorporate these additional terms in a reliable and systematic way. Nevertheless, the hamiltonian analysis does not suffer from these problems, and has the added advantage that also the Gribov problem was analysed in the hamiltonian framework [6, 24, 29] so that the finite-volume analysis is now on a much firmer footing, giving additional support for the reliability of the comparison between the analytic and Monte Carlo results [4]. Due to the essentially non-perturbative nature of the dynamics involved, we believe that it has been worthwhile the effort.

The most important conclusion of this paper is that the effective hamiltonian we have used in the past (apart from a unitary transformation that includes a necessary field renormalization) is confirmed by the hamiltonian analysis. For two-loops it has been a fortunate intuition concerning the absence of "non-local" contributions [12] that (unknowingly) led us to the correct result. For the one-loop $O(g_0^4)$ terms it is because we had neglected in the past terms of the form $g_0^4 c^2 e^2$ (c is the zero-momentum component of the gauge field and e its conjugate momentum), that allowed us to find a unitary transformation which brings the effective hamiltonian to the form we previously [6] employed, but furthermore it gives (see sect. 8) the precise form of these neglected terms (we have good reasons to believe that their influence on the mass ratios is not larger than one percent).

8. Summary

This section will summarise the results for the full continuum effective hamiltonian (eq. (3.47)) and lattice effective lagrangian (eq. (5.1)) for pure SU(2) gauge theory in a finite cubic volume. It will also provide the numerical values of the various coefficients and review the appropriate choice of the boundary conditions (in configuration space) for the wavefunctions. The effective hamiltonian is expressed in terms of the coordinates c_i^a, where $i = \{1, 2, 3\}$ is the spatial index ($c_0 = 0$) and $a = \{1, 2, 3\}$ is the SU(2)-colour index. These coordinates are related to the zero-momentum gauge fields through

$$A_i^a(x) = c_i^a / L \,. \tag{8.1}$$

The following two composite fields will occur

$$F^a_{ij} = -\varepsilon_{abd} c^b_i c^d_j , \qquad r_i = \sqrt{\sum_a c^a_i c^a_i} , \tag{8.2}$$

which represent the field strength and the gauge-invariant "radial" coordinate. The latter will play a crucial role in specifying the boundary conditions. For dimensional reasons the effective hamiltonian is proportional to $1/L$, it will furthermore depend on L through the renormalized coupling constant ($g(L)$) at the scale $\mu = 1/L$. To one-loop order one has (for small L)

$$\frac{1}{g(L)^2} = \frac{-11 \ln(\Lambda_{MS} L)}{12\pi^2} . \tag{8.3}$$

In practice the perturbative expansion of $g(L)$ is known to be reasonable only for unreasonably small volumes. One expresses the masses and the size of the finite volume in dimensionless quantities, like mass ratios and the parameter $z = mL$. In this way, the explicit dependence of g on L is irrelevant. This mode of presenting the physical quantities has turned out to be extremely fruitful [3, 6]. We now give the effective hamiltonian (eq. (3.47)).

$$\begin{aligned} L \cdot H_{\text{eff}}(c) = &-\frac{g^2}{2(1+\alpha_1 g^2)} \sum_{i,a} \frac{\partial^2}{\partial c^{a2}_i} + \frac{1}{4}\left(\frac{1}{g^2} + \alpha_2\right) \sum_{ij,a} F^{a2}_{ij} \\ &+ \gamma_1 \sum_i r_i^2 + \gamma_2 \sum_i r_i^4 + \gamma_3 \sum_{i>j} r_i^2 r_j^2 + \gamma_4 \sum_i r_i^6 + \gamma_5 \sum_{i\neq j} r_i^2 r_j^4 + \gamma_6 \prod_i r_i^2 \\ &+ \alpha_3 \sum_{ijk,a} r_i^2 F^{a2}_{jk} + \alpha_4 \sum_{ij,a} r_i^2 F^{a2}_{ij} + \alpha_5 \det{}^2 c - \frac{g^4}{2} \sum_{ab} \left(\beta_1 \sum_{i\neq j} \left\{ c^a_i c^a_i , \frac{\partial^2}{\partial c^b_j \partial c^b_j} \right\} \right. \\ &+ \beta_2 \sum_{i\neq j} \left\{ c^a_i c^b_j , \frac{\partial^2}{\partial c^a_i \partial c^b_j} \right\} + \beta_3 \sum_i \left\{ c^a_i c^a_i , \frac{\partial^2}{\partial c^b_i \partial c^b_i} \right\} + \beta_4 \sum_i \left\{ c^a_i c^b_i , \frac{\partial^2}{\partial c^a_i \partial c^b_i} \right\} \\ &\left. + \beta_5 \sum_{i\neq j} \left\{ c^a_i c^a_j , \frac{\partial^2}{\partial c^b_i \partial c^b_j} \right\} + \beta_6 \sum_{i\neq j} \left\{ c^a_i c^b_j , \frac{\partial^2}{\partial c^b_i \partial c^a_j} \right\} + \beta_7 \sum_{i\neq j} \left\{ c^a_i c^b_i , \frac{\partial^2}{\partial c^a_j \partial c^b_j} \right\} \right) . \end{aligned} \tag{8.4}$$

We have organized the terms according to the importance of their contributions. The first line gives (when ignoring $\alpha_{1,2}$) the lowest-order effective hamiltonian (eq. (1.1)), whose energy eigenvalues are $O(g^{2/3})$, as can be seen by rescaling c with $g^{2/3}$. Thus, in a perturbative expansion $c = O(g^{2/3})$. The second line includes the "vacuum-valley" effective potential (i.e. the part that does not vanish on the set of abelian configurations). These two lines are sufficient to obtain the mass ratios to an accuracy of better than 5%. The third line gives terms of $O(g^4)$ in the effective

TABLE 4

The numerical values for the coefficients α_i, β_i and γ_i as occurring in the effective hamiltonian of eq. (8.4). The square brackets indicate that these terms are of higher order than $O(g^4)$. Note that $\alpha_{1,2} = \lim_{d\to 3} \alpha^{R}_{1,2}(d)$ (see eq. (4.49))

$\beta_1 = 1.6277104\,E-4$	$\gamma_1 = -3.0104661\,E-1 \times (1+(g/2\pi)^2)$	$\alpha_1 = 2.1810429\,E-2$
$\beta_2 = -1.7090842\,E-4$	$\gamma_2 = -1.4488847\,E-3[-9.9096768\,E-3(g/2\pi)^2]$	$\alpha_2 = 7.5714590\,E-3$
$\beta_3 = 5.3548699\,E-4$	$\gamma_3 = 1.2790086\,E-2[+3.6765224\,E-2(g/2\pi)^2]$	$\alpha_3 = 1.1130266\,E-4$
$\beta_4 = 8.9375854\,E-4$	$\gamma_4 = 4.9676959\,E-5[+5.2925358\,E-5(g/2\pi)^2]$	$\alpha_4 = -2.1475176\,E-4$
$\beta_5 = 6.7878476\,E-4$	$\gamma_5 = -5.5172502\,E-5[+1.8496841\,E-4(g/2\pi)^2]$	$\alpha_5 = -1.2775652\,E-3$
$\beta_6 = 5.3557697\,E-4$	$\gamma_6 = -1.2423581\,E-3[-5.7110724\,E-3(g/2\pi)^2]$	
$\beta_7 = 1.1979070\,E-3$		

potential, hat vanish along the vacuum valley. The remaining terms (the last three lines) have been computed in the present paper. Taken together, this is up to $O(g^4)$ the complete effective hamiltonian. In table 4 we give the numerical values of the various coefficients*. Where numbers are quoted within square brackets, they are of higher order than $O(g^4)$ and were not checked with hamiltonian perturbation theory. Also, it is maybe worthwhile pointing out that we have applied a unitary transformation to the effective hamiltonian as derived from hamiltonian perturbation theory, such that for $\beta_i = 0$ the result coincides with those based on a background field analysis [6]. This is allowed as long as we are only interested in the spectrum of the theory.

To present the lattice effective hamiltonian will not be very informative because its coefficients are too complicated to be presented in print. They can, however, be requested from the author in the form of a C-programme subroutine. Instead we will summarize the lattice effective lagrangian for a cubic spatial lattice of N sites in each spatial direction.

$$\begin{aligned} N\cdot L_{\text{eff}} &= 2N^4\left(\frac{1}{g_0^2}+\alpha_1(N)\right)\sum_i \text{Tr}\left(1-U_i(t+1)U_i^\dagger(t)\right) \\ &+\frac{N^4}{g_0^2}\sum_{ij}\text{Tr}\left(1-U_i(t)U_j(t)U_i^\dagger(t)U_j^\dagger(t)\right)+\gamma_1(N)\sum_i r_i^2 \\ &+\gamma_2(N)\sum_i r_i^4+\gamma_3(N)\sum_{i>j} r_i^2r_j^2+\gamma_4(N)\sum_i r_i^6+\gamma_5(N)\sum_{i\neq j} r_i^2r_j^4 \\ &+\gamma_6(N)\prod_i r_i^2+\frac{\alpha_2(N)}{4}\sum_{ij,a}F_{ij}^{a2}+\alpha_3(N)\sum_{ijk,a} r_i^2F_{jk}^{a2} \\ &+\alpha_4(N)\sum_{ij,a} r_i^2F_{ij}^{a2}+\alpha_5(N)\det{}^2 c\,. \end{aligned} \tag{8.5}$$

* When comparing with the publications predating ref. [4a], one should be aware that α_3 was listed with the wrong sign (e.g. $\kappa_8 = \alpha_3$ in table 1 of ref. [6] should flip sign) and that in table 1 of ref. [4a] α_2 should be divided by 10. We intend to avoid any such errors here.

Here U_i is an SU(2) matrix which is related to the zero-momentum gauge field c_i^a by

$$U_i = \cos\left(\frac{r_i}{2N}\right) + i\sum_a \frac{c_i^a \sigma_a}{r_i} \sin\left(\frac{r_i}{2N}\right), \tag{8.6}$$

where σ_a are the Pauli matrices. This implicitly determines r_i (since we will only need $0 \leqslant r_i \leqslant \pi$, there is no ambiguity). Furthermore, the identities

$$\sum_a (F_{ij}^a)^2 = \frac{r_i^2 r_j^2}{\sin^2\left(\frac{r_i}{2N}\right)\sin^2\left(\frac{r_j}{2N}\right)} \mathrm{Tr}(1 - U_i U_j U_i^\dagger U_j^\dagger), \qquad c_i^a = -i\frac{r_i}{2\sin\left(\frac{r_i}{2N}\right)} \mathrm{Tr}(U_i \sigma_a), \tag{8.7}$$

allow one to express this effective lagrangian entirely in terms of the variables U_i. Table 5 will present numerical results for the coefficients at a few values of N (see table 3 for general N). The perturbative expansion of the effective theory is thus given by the path integral

$$Z = \int \prod_t \mathrm{d}U_i \exp\left[-\sum_t L_{\mathrm{eff}}(U_i(t))\right]. \tag{8.8}$$

Note that time is discrete, as on the original lattice. This partition function therefore defines a transfer matrix whose logarithm gives the lattice effective hamiltonian, see sect. 5 for details. To describe the non-perturbative dynamics in intermediate volumes one has to incorporate the fact that the zero-momentum configuration space is compact. In the hamiltonian analysis this is achieved by imposing appropriate boundary conditions at $r_i = \pi$. An alternative formulation for the lattice effective theory will be discussed at the end of this section.

First we will review the choice of boundary conditions [6, 7], associated to each of the irreducible representations of the cubic group O(3, $\mathbb{Z}$) and to the states that carry electric flux [8]. The best way to describe these boundary conditions, is to observe that the cubic group is the semi-direct product of the group of coordinate permutations S_3 and the group of coordinate reflections Z_2^3. We denote the parity under the coordinate reflection $c_i^a \to -c_i^a$ by $p_i = \pm 1$. The electric flux quantum number for this direction will be denoted by $q_j = \pm 1$. This is related to the more usual [6, 8] additive (mod 2) quantum number e_i by $q_j = \exp(i\pi e_j)$. Note that for SU(2) the electric flux is invariant under coordinate reflections. If not all of the electric fluxes are identical the cubic group is broken to $S_2 \times Z_2^3$, where S_2 ($\sim Z_2$) corresponds to interchanging the two directions with identical electric flux (unequal to the other electric flux). If all the electric fluxes are equal, the wavefunctions are irreducible representations of the cubic group. These are the four singlets

TABLE 5

The numerical values for the coefficients $\alpha_i(N)$ and $\gamma_i(N)$ as occurring in the effective lagrangian of eq. (8.5) for a few values of N (see table 3 for arbitrary N). In this table $\alpha^R_{1,2}(N) = \alpha_{1,2}(N) + (11/12\pi^2)\ln(N\Lambda_L/\Lambda_{MS})$ (eq. (4.48)), such that all coefficients approach their continuum values (table 4) for $N \to \infty$

N	4	6	8	12	16	64
α_1^R	1.6057994 E − 2	1.8164466 E − 2	1.9565022 E − 2	2.0747350 E − 2	2.1197092 E − 2	2.1770191 E − 2
α_2^R	2.3501546 E − 2	1.5958742 E − 2	1.2503792 E − 2	9.8205794 E − 3	8.8467930 E − 3	7.6517996 E − 3
α_3	4.4862048 E − 5	2.6501915 E − 5	5.8501844 E − 5	8.4633735 E − 5	9.4989081 E − 5	1.0987650 E − 4
α_4	1.0170216 E − 3	4.0551245 E − 4	1.6886419 E − 4	−2.1325947 E − 5	−9.6672839 E − 5	−2.0456455 E − 4
α_5	−1.5062536 E − 3	−1.3865466 E − 3	−1.3423463 E − 3	−1.3090597 E − 3	−1.2965637 E − 3	−1.2791963 E − 3
γ_1	−2.0469275 E − 1	−2.6682979 E − 1	−2.8378762 E − 1	−2.9385417 E − 1	−2.9707846 E − 1	−3.0080399 E − 1
γ_2	−6.1626275 E − 3	−2.7698602 E − 3	−2.0167070 E − 3	−1.6585253 E − 3	−1.5597917 E − 3	−1.4553263 E − 3
γ_3	1.3592796 E − 2	1.3367387 E − 2	1.3153548 E − 2	1.2963397 E − 2	1.2889865 E − 2	1.2796495 E − 2
γ_4	2.3009725 E − 4	1.0531806 E − 4	7.5653205 E − 5	5.9991638 E − 5	5.5286176 E − 5	5.0014412 E − 5
γ_5	−1.0464354 E − 4	−8.6613757 E − 5	−7.4108326 E − 5	−6.3915361 E − 5	−6.0149866 E − 5	−5.5487920 E − 5
γ_6	−1.5264054 E − 3	−1.3692563 E − 3	−1.3133879 E − 3	−1.2737564 E − 3	−1.2599797 E − 3	−1.2434562 E − 3

$A^{\pm}_{1(2)}$, which are completely (anti-)symmetric with respect to S_3 and have each of the parities $p_i = \pm 1$. Then there are two doublets $E^{\pm}$, also with each of the parities $p_i = \pm 1$ and finally one has four triplets $T^{\pm}_{1(2)}$. Each of these triplet states can be decomposed into eigenstates of the coordinate reflections. Explicitly, for $T^{\pm}_{1(2)}$ we have one state that is (anti-)symmetric under interchanging the two- and three-directions, with $p_2 = p_3 = -p_1 = \mp 1$. The other two states are obtained through cyclic permutation of the coordinates. Thus, any eigenfunction of the effective hamiltonian with specific electric flux quantum numbers q_i can be chosen to be an eigenstate of the parity operators p_i. The boundary conditions of these eigenfunctions $\Psi_{q,p}(c)$ are simply given by

$$\Psi_{q,p}(c)|_{r_i=\pi} = 0, \qquad \text{if } p_i q_i = -1,$$

$$\frac{\partial}{\partial r_i}(r_i \Psi_{q,p}(c))|_{r_i=\pi} = 0, \qquad \text{if } p_i q_i = +1, \tag{8.9}$$

and one easily shows that with these boundary conditions the hamiltonian is hermitian with respect to the inner product

$$\langle \Psi, \Psi' \rangle = \int_{r_i \leqslant \pi} \mathrm{d}^9 c\, \Psi^*(c) \Psi'(c). \tag{8.10}$$

For negative parity states this description is, however, not accurate [7]. Including those would require a much more detailed analysis [24]. Also note that for T_2 the boundary conditions stated in ref. [6] effectively correspond to a state with two units of electric flux [7] which was later called T^{+}_{11} (see ref. [27] and [3a]).

The boundary conditions arise due to the topological non-trivial nature of configuration space, which forces one to formulate the theory on different coordinate patches (for SU(2) eight in total). There is a nice trick [27] which avoids using boundary conditions. In essence it is based on the fact that the boundary conditions imply [3a] that we formulate the theory on S^3_3. Each three-sphere is associated with one r_i. The equator corresponds with $r_i = \pi$ and it divides the three-sphere in the two-coordinate patches. Since S_3 can naturally be identified with SU(2), we can choose our coordinates to be labeled by an SU(2) matrix V_i. The two coordinate patches are distinguished by sign(Tr(V_i)). The relation between the variables V_i and c^a_i is almost identical to what we found for the lattice in eq. (8.6)

$$V_i = \cos\left(\frac{r_i}{2}\right) + i \sum_a \frac{c^a_i \sigma_a}{r_i} \sin\left(\frac{r_i}{2}\right). \tag{8.11}$$

and one easily verifies that $V_i = U_i^N$. One coordinate patch is defined by $0 \leqslant r_i \leqslant \pi$, for which the relation between U_i and V_i can be inverted uniquely. By defining

$$U_i = \{\mathrm{sign}(\mathrm{Tr}(V_i))V_i\}^{1/N}, \tag{8.12}$$

both coordinate patches of V_i are mapped on $0 \leqslant r_i \leqslant \pi$. Since V_i continuously matches the two coordinate patches, it can be shown ([27b]) (see also [3a]) that the boundary conditions are automatically and correctly implemented, provided the integration measures for the two coordinate patches match continuously at the equator of the three-sphere. Since

$$N\,\mathrm{d}U_i = \left\{\frac{4 - \mathrm{Tr}(U_i)^2}{4 - \mathrm{Tr}(V_i)^2}\right\}\mathrm{d}V_i, \tag{8.13}$$

this is easily seen to be satisfied. The effective lattice theory including the intermediate volume non-perturbative behaviour can now be formulated in terms of an SU(2) lattice gauge theory for a lattice that has only one site in each of the three spatial directions

$$Z = \int \prod_{i,t} \mathrm{d}V_i(t) \exp\left[-\sum_t L'_{\mathrm{eff}}(U_i\{V_i(t)\})\right],$$

$$L'_{\mathrm{eff}}(U_i) = L_{\mathrm{eff}}(U_i) + \sum_i \left\{\ln\left(4 - \mathrm{Tr}(U_i^N)^2\right) - \ln\left(4 - \mathrm{Tr}(U_i)^2\right)\right\}. \tag{8.14}$$

This way of formulating the lattice effective theory allows one to use the Monte Carlo method to measure the masses of the various states. Michael [27b] provides a list of operators with the appropriate quantum numbers. Since the number of degrees of freedom is drastically reduced, it should not be difficult to acquire a reasonable numerical accuracy with modest computational effort. In this way one can avoid the complicated computation of sect. 5 for the logarithm of the transfer matrix.

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